# CS E6204 Lecture 4
# Algorithm for a Genus Distribution of 3-Regular Outerplanar Graphs

**Abstract**

We present a quadratic-time algorithm for calculating the sequence of numbers $g_0$, $g_1$, $g_2$, ... of topologically distinct ways to draw a 3-regular *outerplanar graph* $G$ on each of the respective orientable surfaces $S_0$, $S_1$, $S_2$, ... (without edge-crossings). The total number of ways over all surfaces is $2^n$, where $n$ is the number of vertices of $G$. The key algorithmic features are a characterization of 3-regular outerplanar graphs in terms of *plane trees* and a subsequent synthesis of the graphs by sequences of *amalgamations* of building-block graphs according to *post-order traversals* of those plane trees.



* This lecture is based on a recent research paper by J. L. Gross.

# 1 Introduction

### Outerplanar and outerplane graphs

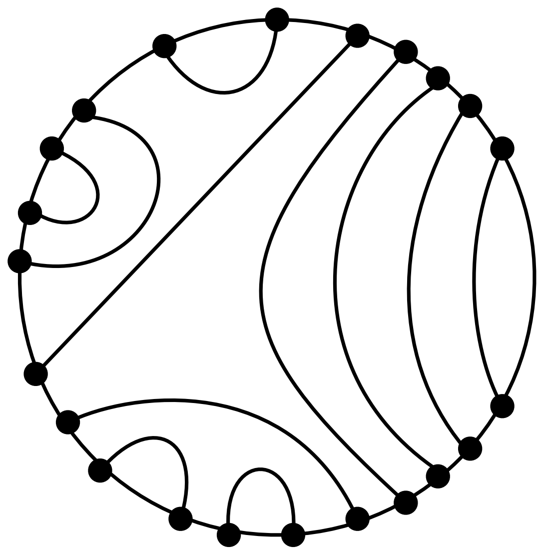


Figure 1.1: A 3-regular outerplane graph.

NOTATION # imbeddings $G \to S_i$ is denoted $g_i(G)$.

DEF. The ***genus distribution*** for graph $G$ is the sequence

$$\{g_i(G)\}$$

### Reading

Genus distribution was first studied in [GF87], [FGS89], and [GRT89]. Recent calculations of the genus distribution of graph amalgamations for recursively defined families of graphs appear in [GKP10], [Gr10a], [KPG10], [?], [PKG10b], and [Gr10a].

Background in topological graph theory appears in [GrTu87] and [BWGT09]. ([MT01] and [Wh01] are alternative sources.)

### Rotation systems (review)

DEF. Two ***equivalent orientable imbeddings*** of a graph $G$ have the same rotation at every vertex of $G$.

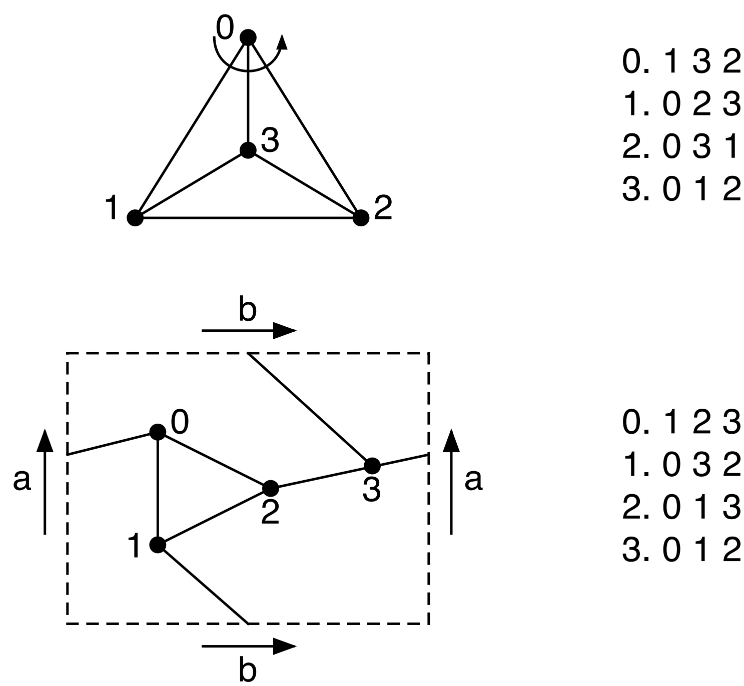



Figure 1.2: Two inequivalent rotation systems for $K_4$.

**Example 1.1** Imbeddings of the complete graph $K_4$.

- 2 in $S_0$ with four 3-gons, like top drawing
- 8 in $S_1$ with 3-gon and 9-gon, like bottom drawing
- 6 in $S_1$ with a 4-gon and an 8-gon

Thus, the genus distribution of $K_4$ is

$$g_0(K_4) \;=\; 2 \qquad g_1(K_4) \;=\; 14$$

## Origin: Graph Isomorphism Problem

Q1: Can two 3-connected non-isomorphic graphs have the same genus distribution? (an "iso-generic" pair)

Q2: If not, how much sampling is needed to distinguish them with probability $p$. (a nearly iso-generic pair)

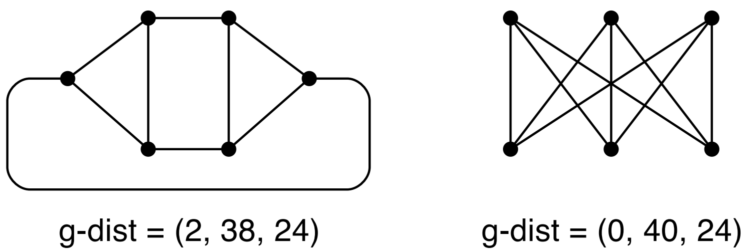



Figure 1.3: Two non-isomorphic graphs.

However, suppose each of there graphs is suspended from a new vertex. Then the resulting genus distributions are

$$\begin{aligned}\gamma\delta(RL_2 + u) &= 0 \quad 884 \quad 129150 \quad 2036086 \quad 3432600 \\ \gamma\delta(K_{3,3} + v) &= 0 \quad 588 \quad 110148 \quad 1973232 \quad 3514752\end{aligned}$$

## Review: Two Basic Results

**Prop 1.1** *For any graph $G$,*

$$\sum_{i \geq 0} g_i(G) \;=\; \prod_{v \in V(G)} ((deg(v) - 1)!)$$

**Thm 1.2** *The minimum-genus problem is NP-complete.*

## Research Problem – Pot of Gold

Is there a sequence of graph operations with the following property: Given an iso-generic pair or a nearly iso-generic pair, the genus distributions of the pairs obtained by the sequence become progressively easier to distinguish statistically?

## 2 Characterizing Cubic Outerplane Graphs

A ***plane tree*** is a rooted tree such that at each vertex, there is a linear ordering of the children.

**Prop 2.1** *Every cubic outerplane graph $G \to \mathbb{R}^2$ can be obtained by adding non-intersecting inner chords to a cycle in the plane.*

**Proof** See Figure 2.1 (left). $\diamondsuit$

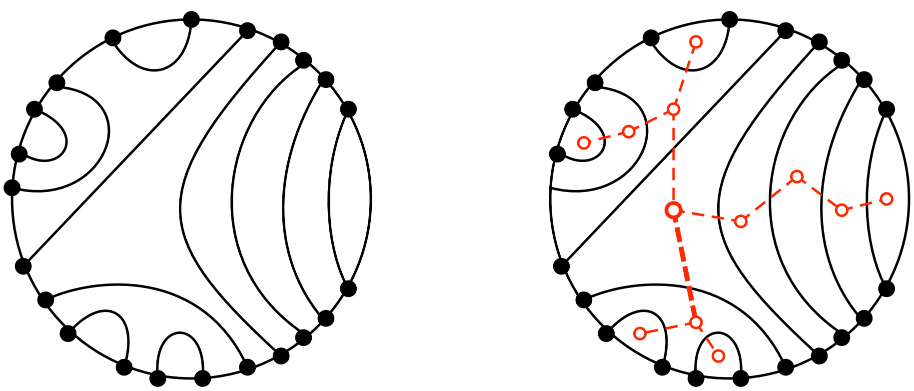
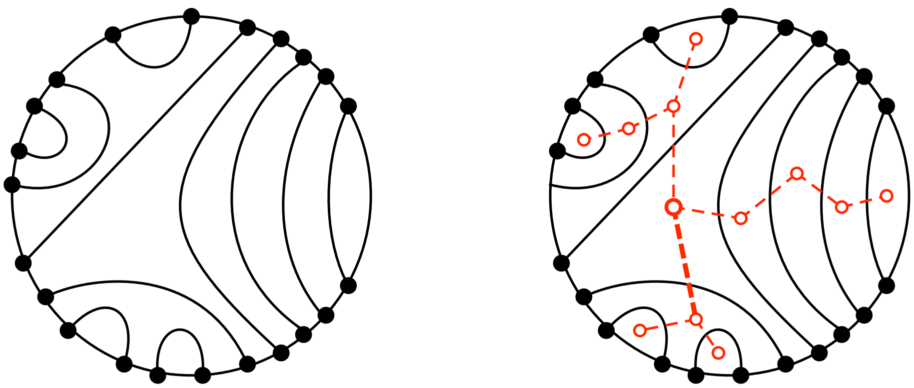


Figure 2.1: An outerplane graph and a characteristic tree.



**Prop 2.2** *The dual of a cubic outerplane graph is the join of a plane tree to a vertex in the exterior region.*

**Proof** We select an arbitrary dual vertex within the outer cycle as a root and an arbitrary child of that root as its leftmost child to make the tree a plane tree, which we call a ***characteristic tree*** – see Fig 2.1 (right) – of the outerplane graph. ♢

We recall (see [AHU83] or [GrYe06]) that the ***post-order*** for a plane tree is obtained from a traversal of the fb-walk for its only face, starting with the edge from the root to its leftmost child. Figure 2.2 assigns integer labels to the vertices of the characteristic tree from Figure 2.1, according to their post-order.

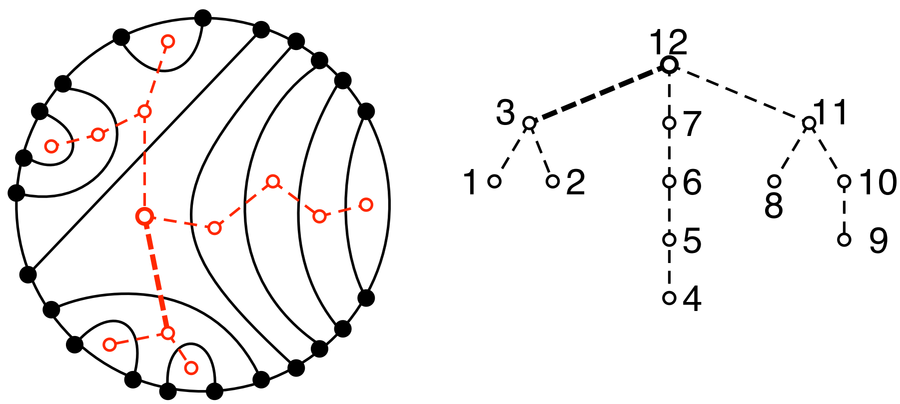
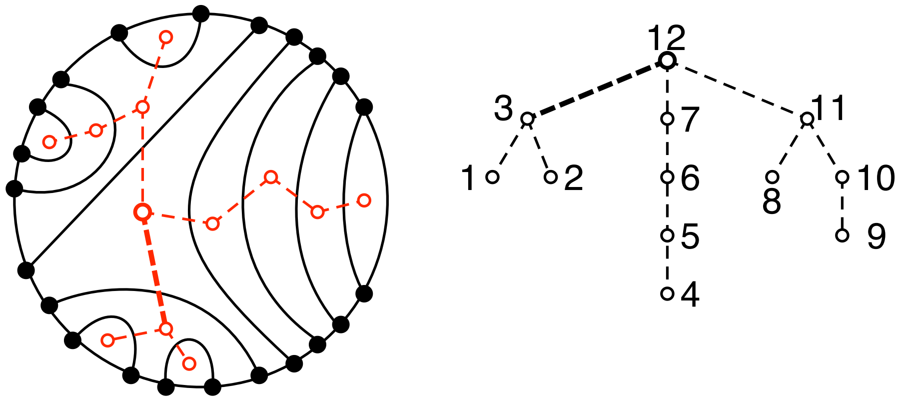



Figure 2.2: Postorder for the characteristic tree.




**Overview of the calculation**

Outline of our calculation plan:

1. Cut the outerplane graph along every chord, so that the chord appears on both sides of the cut as an edge with two 2-valent endpoints.

2. Calculate the genus distribution of each of the graphs resulting from this collection of cuts.

3. Reassemble the graph by an sequence of edge-amalgamations, according to the post-order of the characteristic tree. With each such edge-amalgamation, calculate the resulting genus distribution of the partially reassembled piece.

## 3 Partials and Productions for Edge-Amalgamations

DEF. ***edge-amalgamation*** $(G, d) * (H, e) \; = \; X$

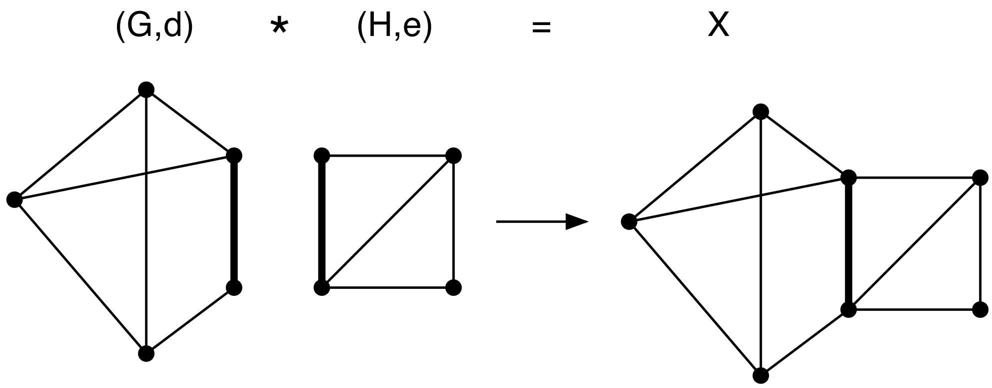
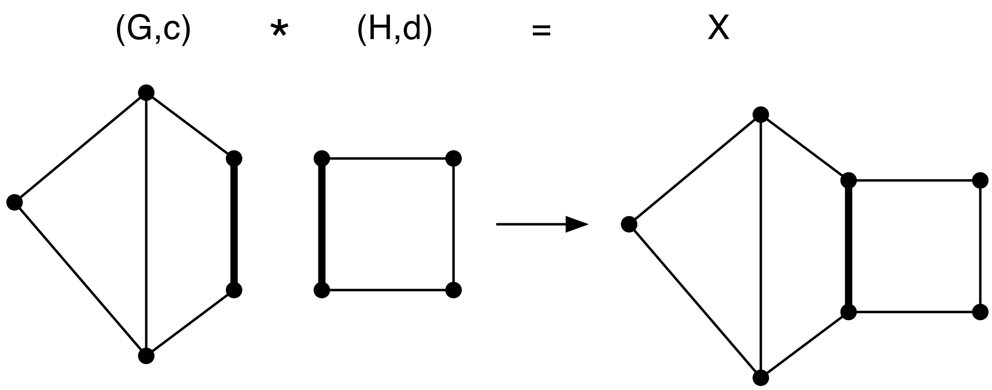



Figure 3.1: Edge-amalgamation of two edge-rooted graphs.

In what follows, we assume

- a given edge-amalgamation is only one of the two possible ways, not both.
- the endpoints of edge-roots $d$ and $e$ are **2-valent**.




**Prop 3.1** *There are exactly four imbeddings of*

$$X = (G, c) * (H, d)$$

*that are consistent with a given pair of rotation systems for* $(G, c)$ *and* $(H, d)$*, respectively. The genera of the four imbedding surfaces depends only on* $\gamma(S_G)$*,* $\gamma(S_H)$*, and the respective numbers of faces in which the two edge-roots* $c$ *and* $d$ *lie.*

**Proof** See [PKG10a]. $\diamondsuit$

## Partial imbedding distributions

We partition the imbeddings of a single-edge-rooted graph $(G, c)$ with $deg(c) = 2$ in a surface of genus $i$ into the subset of ***type-$d_i$ imbeddings***, in which edge-root $c$ lies on two distinct fb-walks, and the subset of ***type-$s_i$ imbeddings***, in which edge-root $c$ occurs twice on the same fb-walk. Moreover, we define

$$
\begin{aligned}
d_i(G,c) &= \text{the number of imbeddings of type-}d_i\text{, and} \\
s_i(G,c) &= \text{the number of imbeddings of type-}s_i.
\end{aligned}
$$

Thus,

$$g_i(G,c) = d_i(G,c) + s_i(G,c)$$

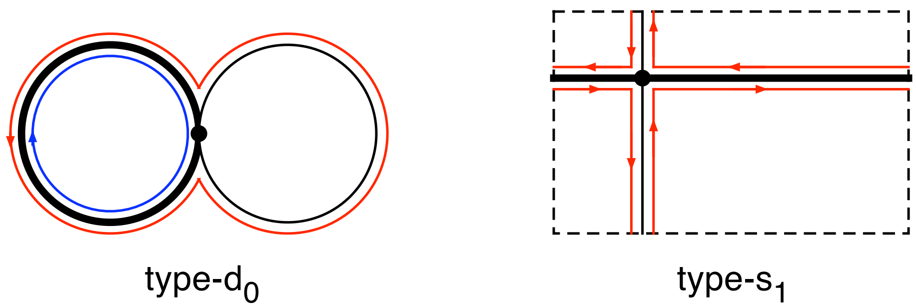



Figure 3.2: the two types of single-root partials.

DEF. The numbers $d_i(G, c)$ and $s_i(G, c)$ are called ***single-root partials***. The sequences

$$\{d_i(G,c) \mid i \geq 0\} \quad \text{and} \quad \{s_i(G,c) \mid i \geq 0\}$$

are called ***partial genus distributions***.

Remark More generally, with a root of higher valence, there would be more partials, corresponding to a larger number of possible configurations of fb-walks at the root.

A double-edge-rooted graph $(H, a, b)$ has many more partials than a single-edge-rooted graph. The two double-root partials of concern here, for the case in which both endpoints of both edge-roots $a$ and $b$ are 2-valent, are as follows:

- The value of the ***double-root partial*** $dd_i''(H, a, b)$ is the number of imbeddings of graph $H$ in the surface $S_i$ such that edge-root $a$ lies on two distinct fb-walks, and there is an occurrence of edge-root $b$ on each of these fb-walks.
- The value of the ***double-root partial*** $ss_i^1$ is the number of imbeddings of graph $H$ in the surface $S_i$ such that both occurrences of edge-root $a$ lie on the same fb-walk, and such that when that fb-walk is broken into two *strands* by deleting the occurrences of edge $a$, one of these strands contains both occurrences of edge-root $b$.

## Productions

A ***production for an edge-amalgamation***

$$(G, c) * (H, d) \;=\; X$$

***of two single-edge-rooted graphs*** is a rule of the form

$$\begin{aligned} p_i(G,t) * q_j(H,u) \;\longrightarrow\; & a_{i+j}\, d_{i+j}(X) \,+\, b_{i+j}\, s_{i+j}(X) \\ & + a_{i+j+1}\, d_{i+j+1}(X) \,+\, b_{i+j+1}\, s_{i+j+1}(X) \end{aligned}$$

where, $p_i$ and $q_j$ are partials, and where $a_{i+j}$, $b_{i+j}$, $a_{i+j+1}$ and $b_{i+j+1}$ are integers. We often write such a rule in the form

$$p_i * q_j \;\longrightarrow\; a_{i+j}\, d_{i+j} \,+\, b_{i+j}\, s_{i+j} \,+\, a_{i+j+1}\, d_{i+j+1} \,+\, b_{i+j+1}\, d_{i+j+1}$$

A ***production for an edge-amalgamation***

$$(G, c) * (H, d, e) \;=\; (X, e)$$

***of a single-edge-rooted graph to a double-edge-rooted graph*** is a similar kind of rule.

Remark A series of fundamental papers ([GKP10], [Gr10a], [KPG10], [PKG10a], [PKG10b], and [Gr10b]) is devoted to calculating productions corresponding to various ways of synthesizing graphs from graphs whose partial genus distributions are known.

Here is the result we need right now.

**Thm 3.2** *Let $(X, f) = (G, d) * (H, e, f)$ be an edge-amalgamation where the endpoints of root-edge $d$ are 2-valent in $G$ and the endpoints of root-edges $e$ and $f$ are 2-valent in $H$. Then the genus distribution of $(X, f)$ conforms to the following productions:*

$$d_i(G, d) * dd''_j(H, e, f) \longrightarrow 2d_{i+j}(X, f) + 2s_{i+j+1}(X) \quad (3.1)$$
$$s_i(G, d) * dd''_j(H, e, f) \longrightarrow 4d_{i+j}(X, f) \quad (3.2)$$
$$d_i(G, d) * ss^1_j(H, e, f) \longrightarrow 4s_{i+j}(X, f) \quad (3.3)$$
$$s_i(G, d) * ss^1_j(H, e, f) \longrightarrow 4s_{i+j}(X, f) \quad (3.4)$$

**Proof** See Theorems 3.1 and 3.2 of [**?**]. ♢

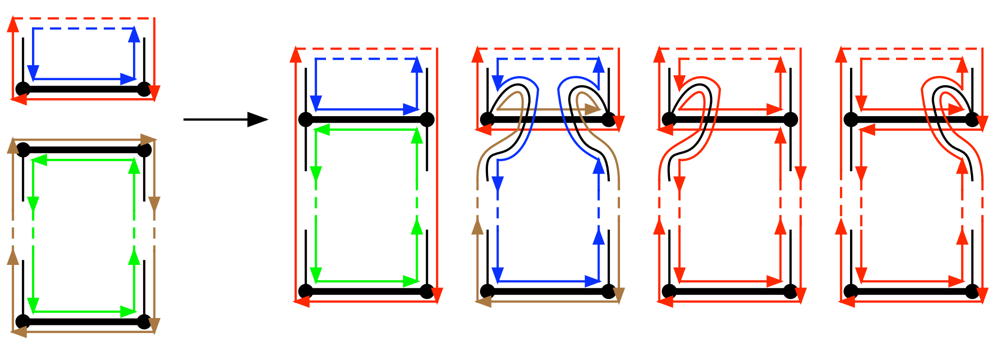


Figure 3.3: Production $d_i(G, d) * dd''_j(H, e, f) \longrightarrow 2d_{i+j}(X, f) + 2s_{i+j+1}(X)$.

In the next subsection, we apply Theorem 3.2 to the example [FGS89] of closed-end ladders, with attention here to the time required for recursive calculation of their genus distributions.

### Calculating the genus distribution of a ladder $L_n$

DEF. ***closed-end ladder*** $L_n$. A few closed-end ladders with edge-roots are shown in Figure 3.4.

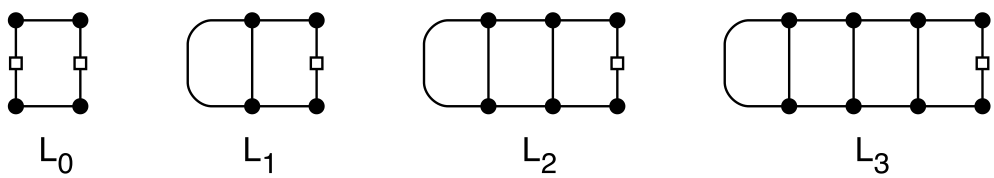
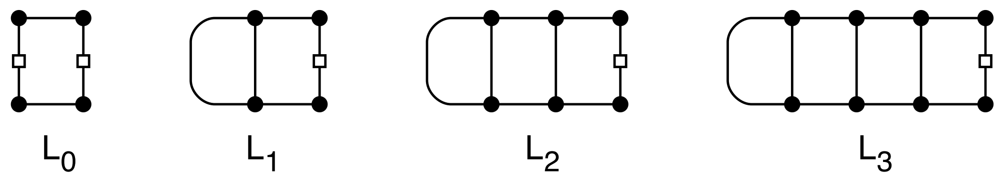



Figure 3.4: Some closed-end ladders, with edge-roots.

For our present purposes, we trisect one of the edges at the end of the ladder and regard the middle sector as the edge-root.




**Recursion basis.** The ladder $L_0$ has the single-root partitioned genus distribution

$$d_0(L_0, w) \;=\; 1$$

and the double-root partitioned genus distribution

$$d_0''(L_0, x, y) \;=\; 1$$

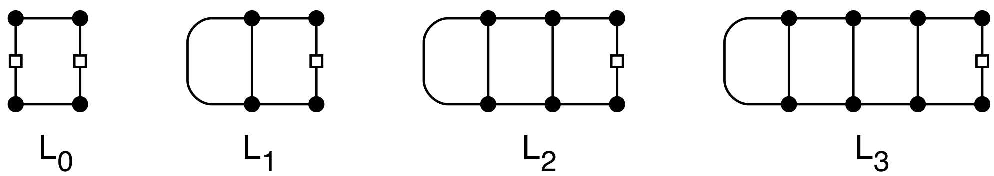
**Reiterated step.** The single-rooted ladder $L_j$ is the edge-amalgamation of a copy of $L_{j-1}$ to a double-rooted copy of $L_0$. For instance, using Production (3.1), we calculate the single-root partitioned genus distribution of the ladder $L_1$:

$$d_0(L_1, x) \;=\; 2 \quad s_1(L_1, x) \;=\; 2$$

Next, using Production (3.1) and Production (3.2), we calculate the single-root partitioned genus distribution of the ladder $L_2$:

$$d_0(L_2, x) \;=\; 4 \quad d_1(L_2, x) \;=\; 8 \quad s_1(L_2, x) \;=\; 4$$

To obtain the single-root partitioned genus distribution of the ladder $L_j$ from the single-root distribution for $L_{j-1}$ and the double-root distribution for $L_0$, Productions (3.1) and (3.2) are sufficient. We continue applying these rules until we obtain a partitioned genus distribution for $L_n$.




**Prop 3.3** *The time needed to calculate the partitioned genus distribution of the ladder $L_n$ is in $O(n^2)$.*

**Proof** In three steps.

1. The number of non-zero partials (over all surfaces $S_i$) of the ladder $L_{n-1}$ is proportional to $n$, and the time needed to apply the relevant production is proportional to the number of non-zero partials.

2. It follows that the time needed to calculate the partials for $L_n$ from the partials for $L_{n-1}$ is proportional to $n$.

3. Accordingly, the time needed to calculate the partials for $L_n$, starting from $L_0$, is proportional to $n^2$. $\diamondsuit$

## 4 Genus Distribution of Star-Ladders

Closed-end ladders are the simplest cubic outerplanar graphs. Their generalization to ***star-ladders*** is a base case of our algorithm for the genus distribution of any cubic outerplanar graph. The star-ladder $SL_{(1,2,3)}$ is shown in Figure 4.1.

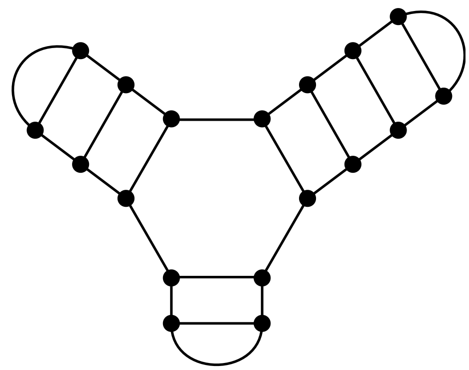
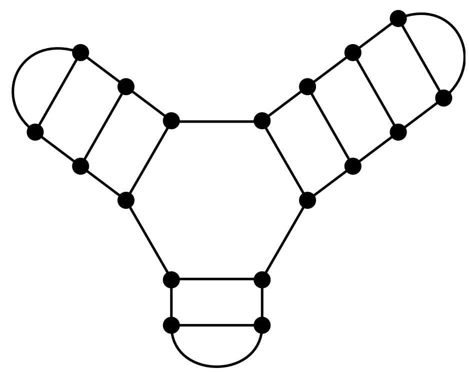


Figure 4.1: The star-ladder $SL_{(3,2,1)}$.

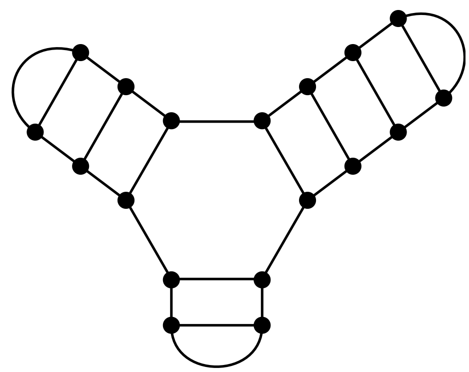
TERMINOLOGY Each of the closed-end ladders is a ***ray*** of the star.

TERMINOLOGY We may refer to any graph homeomorphic to a star-ladder as a star-ladder. That is, a star-ladder remains a star-ladder after one or more edges is subdivided.



**Prop 4.1** *Two star-ladders are isomorphic if the signature of one can be obtained by a rotation and/or a reversal of the signature of the other.*

Remark However, placing an edge-root at the tip of one of the rays may not be equivalent to placing it at the tip of another ray.



### Splitting a single edge-root into a double edge-root

From Figure 4.1 (reproduced above), it appears that one might need $r$ edge-roots to paste on $r$ rays.

**Prop 4.2** *OUCH! is an appropriate reaction to the preceding sentence.*

**Proof** The # of partials increases rapidly with the # of edge-roots. Thus, using more than two roots is formidable. $\diamondsuit$

To avoid the need for having more than two roots at a time, we now introduce a fundamental new method.

Let $(G, a)$ be an edge-rooted graph such that both endpoints of edge $a$ are 2-valent. By ***splitting the root-edge*** $a$ we mean trisecting edge $a$ and regarding the two outer segments as edge-roots of the resulting graph. (We may call one of these outer segments $a$.)

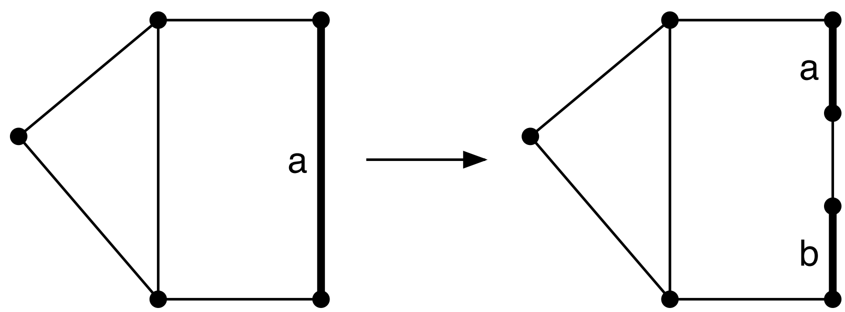



Figure 4.2: Splitting an edge-root.

**Thm 4.3** *Let $(G, a, b)$ be a double-edge-rooted graph such that both endpoints of root-edges $a$ and $b$ are 2-valent, and such that there is a path from $a$ to $b$ along which every internal vertex is 2-valent. Then for every non-negative integer $i$,*

$$dd_i''(G, a, b) = d_i(G, a) \tag{4.1}$$

$$ss_i^1(G, a, b) = s_i(G, a) \tag{4.2}$$

*Moreover, every other double-root partial of $(G, a, b)$ is zero-valued.*

**Proof** See Figure 4.3. ◊

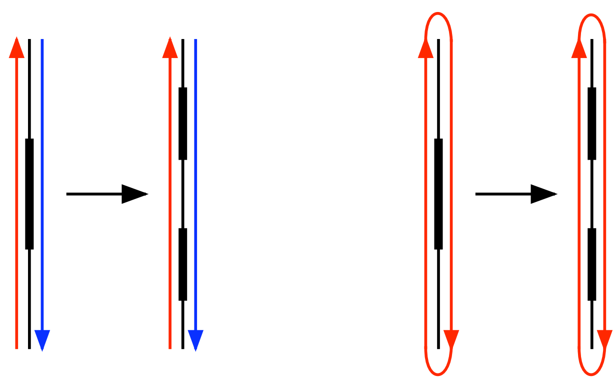


Figure 4.3: Splitting a single-root partial.

### Algorithm for genus distribution of a star-ladder

To calculate the genus distribution of the star-ladder $SL_T$ with signature $T = (k_1, k_2, \ldots, k_r)$.

1. Start by placing an edge-root on each of the edges $e_2$ and $e_4$ of the cycle-graph $C_{2r}$.

2. Amalgamate the ladder $L_{k_1}$ to the cycle graph on edge $e_2$ and calculate the single-edge-root genus distribution for the star-ladder $(SL_{(k_1)}, e_4)$ using the given production rules.

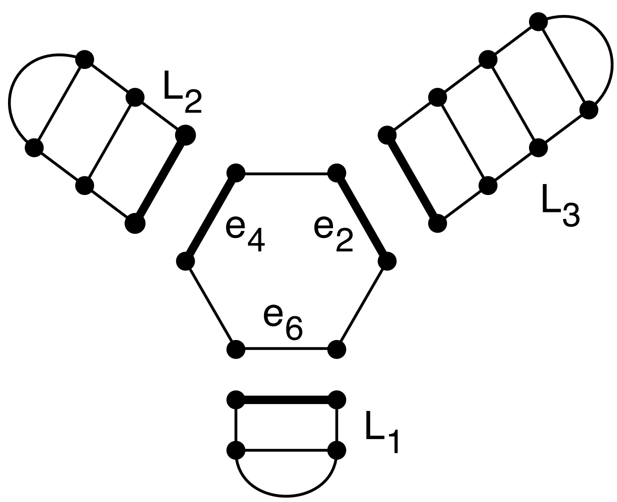
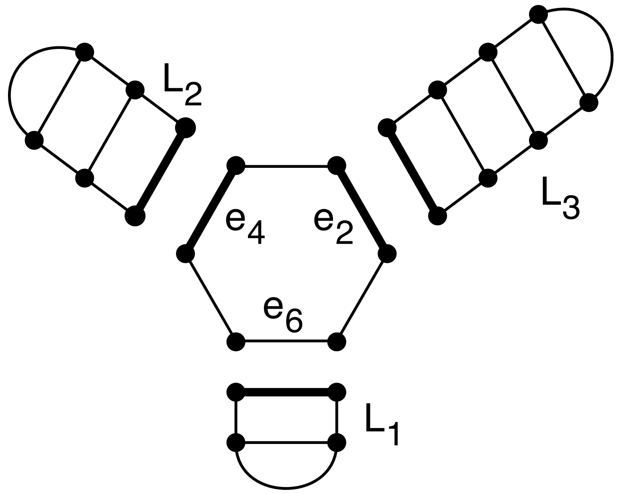



Figure 4.4: Assembling star-ladder $SL_{(3,2,1)}$.




3. Split root-edge $e_4$ into two edge-roots. For this purpose, the other edge-root may be regarded as edge $e_6$. Theorem 4.3 enables us to transform the single-edge distribution for $(SL_{(k_1)}, e_4)$ into the double-root distribution for $(SL_{(k_1)}, e_4, e_6)$.

4. Iterate this process of pasting a ladder across its only root-edge to the growing star-ladder at the "older" of its two edge-roots, and then splitting the remaining edge-root. Continue until all but one of the rays have been pasted onto the star.

5. Build the $r^{\text{th}}$ ray by edge-amalgamating double-rooted copies of $L_0$ outward from the body of the ray, so that we finish with a single edge-root at the tip of the $r^{\text{th}}$ ray, which we will need for what follows in the next section.

**Example 4.1** Consider the star-ladder $(SL_{(0,1,1)}, x)$, where the edge-root $x$ is at the tip of the ray corresponding to 0 in the signature. As shown in Figure 4.5, we take $(X_0, x, y)$ to be a cycle graph with two non-adjacent edges as roots, and we have $(L_1, a)$ as a ladder with the middle sector of an end-rung as its root. Let $(X_2, y)$ be the result of the amalgamation.

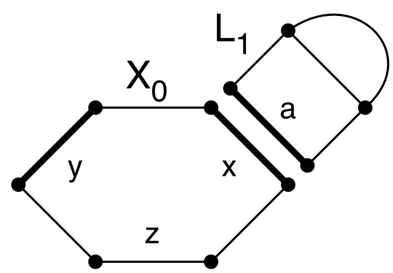



Figure 4.5: Amalgamating ladder $L_1$ to $X_0$.

We start with the partials $d_0(L_1, a) = 2$, $s_1(L_1, a) = 2$, and $dd''_0(X_0, x, y) = 1$ and we apply the productions

$$\begin{aligned} d_i(L_1, a) * dd''_j(X_0, x, y) &\longrightarrow 2d_{i+j}(X_1, y) + 2s_{i+j+1}(X_1, y) \\ s_i(L_1, a) * dd''_j(X_0, x, y) &\longrightarrow 4d_{i+j}(X_1, y) \end{aligned}$$

It follows that

$$\begin{array}{lllll} d_0(X_1) = 2 \cdot 2 \cdot 1 = 4 & d_1(X_1) = 4 \cdot 2 \cdot 1 = 8 \\ & s_1(X_1) = 2 \cdot 2 \cdot 1 = 4 \end{array}$$

We continue with the next ladder.

We split root-edge $y$ and paste another copy of $L_1$ onto $X_1$, as indicated in Figure 4.6.

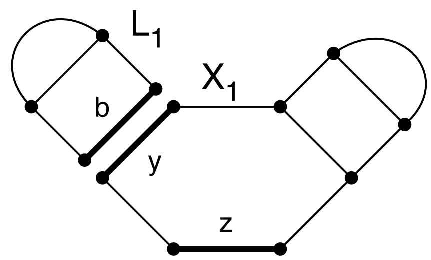



Figure 4.6: Amalgamating ladder $L_1$ to $X_1$.

$$\begin{array}{lll} d_0(L_1,b) = 2 & s_1(L_1,b) = 2 & \\ dd_0(X_1) = 4 & dd_1(X_1) = 8 & ss_1(X_1) = 4 \end{array}$$

We apply the productions

$$\begin{array}{rcl} d_i(L_1,b) * dd''_j(X_1,y,z) & \longrightarrow & 2d_{i+j}(X_2,z) + 2s_{i+j+1}(X_2,z) \\ s_i(L_1,b) * dd''_j(X_1,y,z) & \longrightarrow & 4d_{i+j}(X_2,z) \\ d_i(L_1,b) * ss^1_j(X_1,y,z) & \longrightarrow & 4s_{i+j}(X_2,z) \\ s_i(L_1,b) * ss^1_j(X_1,x,z) & \longrightarrow & 4s_{i+j}(X_2,z) \end{array}$$

It follows that

$$\begin{array}{rcccccc} d_0(X_2) & = & 2d_0dd''_0 & = & 2\cdot 2\cdot 4 & = & 16 \\ d_1(X_2) & = & 2d_0dd''_1 + 4s_1dd''_0 & = & 2\cdot 2\cdot 8 + 4\cdot 2\cdot 4 & = & 64 \\ d_2(X_2) & = & 4s_1dd''_1 & = & 4\cdot 2\cdot 8 & = & 64 \\ s_1(X_2) & = & 2d_0dd''_0 + 4d_0ss^1_1 & = & 2\cdot 2\cdot 4 + 4\cdot 2\cdot 4 & = & 48 \\ s_2(X_2) & = & 2d_0dd''_1 + 4s_1ss^1_1 & = & 2\cdot 2\cdot 8 + 4\cdot 2\cdot 4 & = & 64 \end{array}$$

We complete with the final ladder.

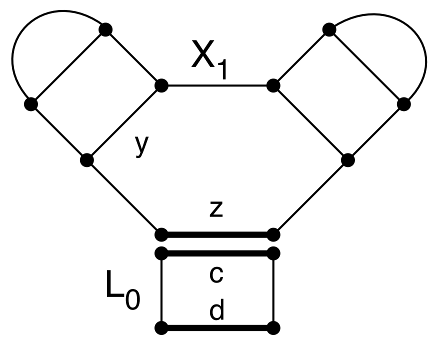



Figure 4.7: Amalgamating ladder $L_0$ to $X_2$.

$$
\begin{array}{rcccccc}
d_0(X_2) & = & 2d_0dd''_0 & = & 2\cdot 2\cdot 4 & = & 16 \\
d_1(X_2) & = & 2d_0dd''_1 + 4s_1dd''_0 & = & 2\cdot 2\cdot 8 + 4\cdot 2\cdot 4 & = & 64 \\
d_2(X_2) & = & 4s_1dd''_1 & = & 4\cdot 2\cdot 8 & = & 64 \\
s_1(X_2) & = & 2d_0dd''_0 + 4d_0ss^1_1 & = & 2\cdot 2\cdot 4 + 4\cdot 2\cdot 4 & = & 48 \\
s_2(X_2) & = & 2d_0dd''_1 + 4s_1ss^1_1 & = & 2\cdot 2\cdot 8 + 4\cdot 2\cdot 4 & = & 64 \\
dd''_0(L_0, c, d) & = & 1 & & & &
\end{array}
$$

We apply the productions

$$
\begin{array}{rcl}
d_i(L_1, b) * dd''_j(X_2, y, z) & \longrightarrow & 2d_{i+j}(X_3, z) + 2s_{i+j+1}(X_3, z) \\
s_i(L_1, b) * dd''_j(X_2, y, z) & \longrightarrow & 4d_{i+j}(X_3, z)
\end{array}
$$

And we conclude

$$
\begin{array}{rcccccc}
d_0(X_3) & = & 2d_0dd''_0 & = & 2\cdot 16\cdot 1 & = & 32 \\
d_1(X_3) & = & 2d_1dd''_0 + 4s_1dd''_0 & = & 2\cdot 64\cdot 1 + 4\cdot 48\cdot 1 & = & 320 \\
d_2(X_2) & = & 2d_2dd''_0 + 4s_2dd''_0 & = & 2\cdot 64\cdot 1 + 4\cdot 64\cdot 1 & = & 384 \\
s_1(X_2) & = & 2d_0dd''_0 & = & 2\cdot 16\cdot 1 & = & 32 \\
s_2(X_2) & = & 2d_0dd''_0 & = & 2\cdot 64\cdot 1 & = & 128 \\
s_3(X_2) & = & 2d_0dd''_0 & = & 2\cdot 64\cdot 1 & = & 128
\end{array}
$$

**Prop 4.4** *The time needed to calculate the partitioned single-root genus distribution of a star-ladder $SL_T$ is in $O(n^2)$, where $n$ is the total number of vertices.*

**Proof** The **number of non-zero partials for each ray** is proportional to the number of vertices in the ray.

Similarly, the **number of non-zero partials for the body of the growing star-ladder** is proportional to its number of vertices. (This is a benefit of bounded degree. More generally, the number of non-zero partials over all surfaces in the genus range grows in proportion to the number of edges.)

Each time a ray is amalgamated to the body of the growing star-ladder, the **time needed to apply the relevant production rules** is proportional to the product of the numbers of non-zero partials in the new ray and in the pre-existing star-ladder.

Since the sum $\sum_{i=1}^{n} i^2$ is proportional to $n^3$, we need to be concerned that the total number of multiplications required as the entire star-ladder is constructed might be of order $n^3$. However, let us suppose that the $i^{\text{th}}$ ray has $k_i$ vertices. Then the total number of multiplications is approximately

$$\begin{aligned}\sum_{j=1}^{r} k_j \sum_{i=1}^{j-1} k_i &= \sum_{j=1}^{r} k_j k_1 + k_j k_2 + \cdots + k_j k_{j-1} \\ &\leq (k_1 + \cdots + k_r)(k_1 + \cdots + k_r) \\ &= n^2\end{aligned}$$

Accordingly, the time needed to calculate the partials for the star-ladder $SL_T$ is in $O(n^2)$. $\diamondsuit$

## 5 Algorithm for a Cubic Outerplanar Graph $G$

**Preliminary steps.** Construct a characteristic tree for $G$ and the post-order for that tree, as in §2.

**Basis step.** If the characteristic tree has only one vertex, then $G$ is a cycle (with no chords), and the genus distribution is

$$d_0(C_n, a) = 1$$

**Ind hyp.** Suppose that this algorithm works for every case in which the characteristic tree has $n - 1$ vertices.

**Ind step.** Consider the case of a characteristic tree with $n$ vertices.

1. Construct the closed-end ladder corresponding to the first vertex $v_1$ in the post-order, with an edge-root at the tip of the ray by which it is to be joined to the body of the star-ladder correponding to its parent; and construct the single-root genus distribution for that closed-end ladder.

2. Also construct the genus distributions for the subgraphs (all cubic outerplanar!) corresponding to the subtrees at each of the siblings (if any) of $v_1$, each with an edge-root at the tip of the ray by which it is joined to the body of the parent star ladder, which is possible, by the induction hypothesis.

3. Then one at a time, according to the post order, double the edge-root of the parent star-ladder and edge-amalgamate each of these sibling subgraphs to the parent. At the surviving edge-root (obtained by doubling the edge-root prior to edge-amalgamating the last subgraph among the siblings), attach double-edge-rooted copies of $L_0$ iteratively

until a ladder of appropriate length is constructed, with an edge-root exactly where needed to edge-amalgamate this subgraph to its parent.

4. Continue until the graph $G$ and its genus distribution are fully constructed.

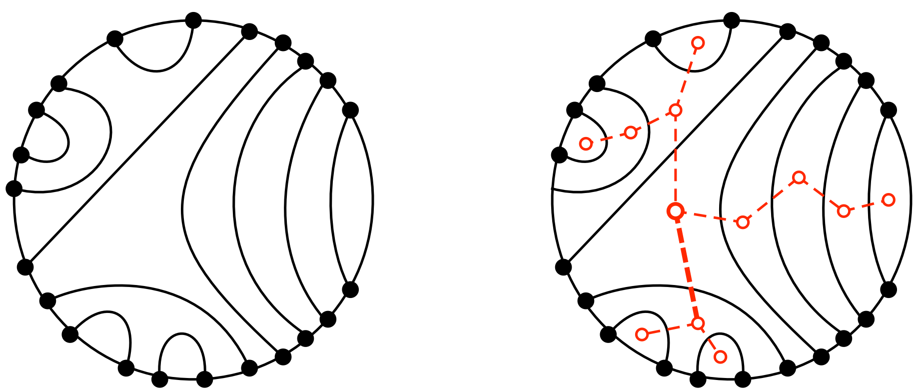


Figure 5.1: An outerplane graph and a characteristic tree.

**Prop 5.1** *The time needed to calculate the genus distribution of a cubic outerplanar graph $G$ is in $O(n^2)$, where $n$ is the total number of vertices.*

**Proof** The proof is similar to that for the star-ladders. $\diamondsuit$

## References

[AHU83] A. V. Aho, J. E. Hopcroft, J. D. Ullman, *Data Structures and Algorithms*, Addison-Wesley, 1983.

[BWGT09] L. W. Beineke, R. J. Wilson, J. L. Gross, and T. W. Tucker (editors), *Topics in Topological Graph Theory*, Cambridge University Press, 2009.

[FGS89] M. L. Furst, J. L. Gross and R. Statman, Genus distribution for two classes of graphs, *J. Combin. Theory (B)* **46** (1989), 22–36.

[Gr10a] J. L. Gross, Genus distribution of graph amalgamations, II. Self-pasting at 2-valent co-roots, preprint, 2009, 21 pages.

[Gr10b] J. L. Gross, Genus distribution of graphs under surgery: adding edges and splitting vertices, *Preprint* (2009), 21 pages.

[GF87] J. L. Gross and M. L. Furst, Hierarchy for imbedding-distribution invariants of a graph, *J. Graph Theory* **11** (1987), 205–220.

[GKP10] J. L. Gross, I. Khan, and M. Poshni, Genus distribution of graph amalgamations, I: Pasting two graphs at 2-valent roots, *Ars Combinatoria* **94** (2010), 33–53.

[GKR93] J. L. Gross, E. W. Klein and R. G. Rieper, On the average genus of a graph, *Graphs and Combinatorics* **9** (1993), 153–162.

[GRT89] J. L. Gross, D. P. Robbins and T. W. Tucker, Genus distributions for bouquets of circles, *J. Combin. Theory (B)* **47** (1989), 292–306.

[GrTu87] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Dover, 2001; original edn. Wiley, 1987).

[GrYe06] J. L. Gross and J. Yellen, *Graph Theory and Its Applications*, Second Edition, CRC Press, 2006.

[KPG10] I. Khan, M. Poshni, and J. L. Gross, Genus distribution of graph amalgamations, III: Pasting when one root has arbitrary degree, preprint 2010, 25 pages.

[KL93] J. H. Kwak and J. Lee, Genus polynomials of dipoles, *Kyungpook Math. J.* **33** (1993), 115–125.

[KL94] J. H. Kwak and J. Lee, Enumeration of graph embeddings, *Discrete Math.* **135** (1994), 129–151.

[Mc87] L. A. McGeoch, *Algorithms for two graph problems: computing maximum-genus imbedding and the two-server problem*, PhD thesis, Carnegie-Mellon University, 1987.

[MT01] B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins University Press, 2001.

[PKG10a] M. Poshni, I. Khan, and J. L. Gross, Genus distribution of graphs under edge amalgamations, *Ars Mathematica Contemporanea* (2010), to appear.

[PKG10b] M. Poshni, I. Khan, and J. L. Gross, Edge-operation effect on genus distribution, II. Self-amalgamation on a pair of edges, *Private Report* (2010), 21 pages.

[St90] S. Stahl, Region distributions of graph embeddings and Stirling numbers, *Discrete Math.* **82** (1990), 57–78.

[St91a] S. Stahl, Permutation-partition pairs III: Embedding distributions of linear families of graphs, *J. Combin. Theory (B)* **52** (1991), 191–218.

[St91b] S. Stahl, Region distributions of some small diameter graphs, *Discrete Math.* **89** (1991), 281–299.

[St95a] S. Stahl, Bounds for the average genus of the vertex amalgamation of graphs, *Discrete Math.* **142** (1995), 235–245.

[St95b] S. Stahl, On the average genus of the random graph, *J. Graph Theory* **20** (1995), 1–18.

[Te00] E. H. Tesar, Genus distribution of Ringel ladders, *Discrete Math.* **216** (2000) 235–252.

[VW07] T. I. Visentin and S. W. Wieler, On the genus distribution of $(p, q, n)$-dipoles, *Electronic J. of Combin.* **14** (2007), Art. No. R12.

[WL06] L. X. Wan and Y. P. Liu, Orientable embedding distributions by genus for certain types of graphs, *Ars Combin.* **79** (2006), 97–105.

[WL08] L. X. Wan and Y. P. Liu, Orientable embedding genus distribution for certain types of graphs, *J. Combin. Theory (B)* **47** (2008), 19–32.

[Wh01] A. T. White, *Graphs of Groups on Surfaces*, Elsevier, 2001.