Comp 131

CLASS NOTES: HANDOUT # 1B
Propositional Logic: Rules for Reasoning

1 Propositional Logic

1.1 Formal Languages, Formal Rules

The validity of some statements and some steps in reasoning seems to have more to do with the *form* than the *content* of the claims or inferences being made. Consider, for example, the following (not terribly exciting) example of reasoning, related to computing earned income credit (EIC) on an income tax form. Premisses (what we are given) are above the line, and the conclusion below.

If Bill is eligible for EIC then Bill is married and has income below 20K
Bill is eligible for EIC

Bill is married

Now consider the following argument:

If Alice can run for office then Alice is over 35 and (Alice is) a US citizen.
Alice can run for office

Alice is over 35

There is something the two arguments have in common, which is really independent of the fact that one is about Bill and the other about Alice, or that one discusses marriage and the other age. The conclusions are a consequence of the **structure** of the premisses, and not of all the detailed information they convey.

In some sense the best way to bring out what steps of logic are really being used here, i.e. what the two arguments have in common is to throw away what is *contingent* in each argument and reduce it to bare bones:

If A holds then B and C both hold
A holds

B must hold

Or, to make it even more bare bones, using $A \to D$ for "if A then D" and the symbol "$\wedge$" for "and"

$$\frac{A \to (B \wedge C) \qquad A}{B} \tag{1}$$

where it is understood that if we are given the facts above the line then we may conclude what is below the line. Note that in some sense the letters A, B and C above are *variables*. The letter "A" doesn't actually *say* anything: it stands for *any statement you want*. Another way of saying this is you may "plug in" or substitute any statement for the letters A, B, C and the rule (1) remains valid. This is like substituting in a number for the letter x in

$$x + x = 2x.$$

But, whereas most of us feel that we know what numbers are, just what is a "statement" in the sense used above?

We need to abstract the notion of *statement* and to provide rules for manipulating them that will allow us to turn elementary steps of reasoning into a mathematical discipline, like algebra. To this end we are going to work with a *formal language*, a language built up from variables or symbols using logical *connectives*, analogous to the $+$ and $\times$ operations of arithmetic.

The reader might wonder at this point why not just let "statements" be sentences in English. There are two reasons. First, a psychological one suggested above, is we are likely to confuse the contingent information with structural information. Also, in order to give a mathematical treatment of the subject, we need to have a precise definition of what a correctly formed statement is. What constitutes a correctly formed statement in English is harder to pin down than one might first imagine. (When we write some programs that read sentences this will become clearer). What we will do is start with basic building blocks called *atomic statements or propositions* and then show how to make *compound statements* using the all important logical connectives **and**, **or**, **implies** and **not**. Atomic statements will be be assumed to lack such connectives. We will just use certain letters for these atomic statements, like $A, B, \ldots$ because *from the standpoint of logic it doesn't matter what they say*.

1.2 Propositions and Connectives

A *proposition* is any statement built up from *atoms* or *atomic propositions* (which might be letters $A, B, C...$, or statements we agree not to analyze *i.e.* not to break up into smaller components), and the logical constant *falsity* (written $\bot$) using the *logical connectives*

and, **or**, **implies** and **not**.

These connectives are also written

$$\wedge \text{ or } \& \quad \text{for} \quad \textbf{and} \tag{2}$$
$$\vee \quad \text{for} \quad \textbf{or} \tag{3}$$
$$\neg \text{ or } \sim \quad \text{for} \quad \textbf{not} \tag{4}$$
$$\rightarrow \text{ or } \supset \quad \text{for} \quad \textbf{implies} \tag{5}$$

We can define what a proposition is *inductively*, that is to say by telling you how to build them from the ground up:

1. If A is an atom then A is a proposition.
2. $\bot$ (falsity) is a proposition.
3. if P,Q are propositions then

   $$(P \textbf{ and } Q), (P \textbf{ or } Q), (P \textbf{ implies } Q), (\textbf{not } P)$$

   are propositions. Writing this same statement symbolically:

   $$(P \wedge Q), (P \vee Q), (P \rightarrow Q), (\neg P)$$

   are propositions.

1.3 Rules, Premisses and Conclusions

A logical *rule* is a figure of the form

$$\frac{P_1 \quad P_2 \quad \cdots \quad P_n}{Q} \qquad \frac{\longleftarrow \text{premisses}}{\longleftarrow \text{conclusion}}.$$

The premisses may consist of one, two or three propositions. The conclusion will always consist of one proposition. The rules can be read in a number of ways:

1. *From the facts* P_1 *and* $P_2 \cdots$ *and* P_n *we can infer the fact* Q, or we may read it *upside-down* as follows:

2. *In order to prove Q we must first prove P_1 and P_2 and $\cdots$ P_N.*

In the next pages we will introduce rules for each logical connective. You should try reading them in both of these ways.

Rules attempt to capture *basic steps of reasoning*. You can view them as the tiniest possible steps that can be taken in an argument. All proofs or arguments are to be built up using these basic steps. You can also look at them as *a way to make precise mathematically what the connectives* $\wedge, \vee, \rightarrow$ *mean.* We now list the basic rules of classical propositional logic. They may seem mystifying at first, but we will devote ample time to them in class.

First the rules for *conjunction* or "and". There are two sets of rules: *and introduction rules*, also called $\wedge_I$, as well as the two *and elimination rules* , called the elim-left and elim-right rules, or $\wedge_{E_\ell}$ and $\wedge_{E_r}$.

and $$\frac{P \quad Q}{P \wedge Q}\,[\wedge_I] \qquad \frac{P \wedge Q}{P}[\wedge_{E_\ell}] \quad \frac{P \wedge Q}{Q}[\wedge_{E_r}]$$

Recall: we may read the left hand rule as asserting:

From the fact P and the fact Q we can infer the fact $P \wedge Q$.

or:

In order to prove $P \wedge Q$ we must first prove P and Q

The right hand rules say: *From the fact $P \wedge Q$ is true we may conclude P is true. We may also conclude Q is true.*

Rules for "implication": These rules tell us when we may conclude a statement of the form $P \rightarrow Q$ is true, and what we can infer from such a statement. These have a new feature: premisses get *cancelled*. The line through $\not{P}$ is called a *cancellation line*. All rules with cancellation contain figures of the form

$$\begin{array}{c} P \\ \vdots \\ Q \end{array}$$

(but with P cancelled) meaning *from P we are able to infer Q*.

A rule with a cancellation should be read as follows. First look at everything above the line, *without cancellations*, and read *if we can infer this... then we can infer what is below the line just from the premisses remaining uncancelled.* For example The first rule for $\rightarrow$ (implication introduction or $\rightarrow_I$) looks like this:

implication introduction $$\frac{\begin{array}{c} \not{P} \\ \vdots \\ Q \end{array}}{P \rightarrow Q}[\rightarrow_I]$$

It can be read as follows.

if you can infer Q from the assumption P then you can prove $P \rightarrow Q$ without assuming P.

If you read it upside down, as suggested earlier, it reads:

In order to prove $P \rightarrow Q$ you should show that Q can be inferred from the assumption P.

Yet another way to think about cancellation is as a "hypothetical inference" or a *record of a thought experiment* in which you say: let's "pretend" that P holds, and suppose that I can then infer Q. Then what do I know (whether or not P holds)? I know that $P \rightarrow Q$. Here's a somewhat fanciful example.

Consider the statement: "if the sun became a supernova, the atmosphere would be blown away." How does one establish this? By *assuming* that the conditions of a supernova sun actually hold in our mathematical models and observing the consequences for the atmosphere. After we have carried out this thought experiment, we can take a deep breath without fear. The atmosphere is still there, because the sun is still its usual self. The premiss "The sun is supernova" is *not* true. The conclusion "the earth has no atmosphere" is *not* true. All that *is* true is that we now know the implication:

If the sun became a supernova the earth would have no atmosphere

i.e., in stages:

1. The thought experiment

$$\begin{array}{c} \text{the sun is a supernova} \\ \vdots \\ \text{the earth has no atmosphere} \end{array} \qquad \left\{ \begin{array}{l} \text{by some complicated chain of} \\ \text{deduction using logic (and the} \\ \text{laws of physics)} \end{array} \right.$$

2. the conclusion

$$\dfrac{\begin{array}{c} \cancel{\text{the sun is a supernova}} \\ \vdots \\ \text{the earth has no atmosphere} \end{array}}{\text{the sun is a supernova } \rightarrow \text{ the earth has no atmosphere}}$$

Here are both the introduction and elimination rules for $\rightarrow$, just so you can see them side by side.

$$\textbf{implication} \qquad \dfrac{\begin{array}{c} \cancel{P} \\ \vdots \\ Q \end{array}}{P \rightarrow Q}[\rightarrow_I] \qquad \dfrac{P \rightarrow Q \quad P}{Q}[\rightarrow_E]$$

The second rule (implication elimination or $\rightarrow_E$) says:

> *From $P \rightarrow Q$ and P you may infer Q. Or: from the fact that P implies Q and the fact that P is true you may infer Q.*

This is the opposite of our thought experiment! This says if the premiss P should actually be true then Q will be true!

Rules for "negation": These rules will generate a good deal of discussion in class! The first one, $\perp$-elimination, says, from $\perp$ you may infer anything. It's a blank check. The meaning is that if you can derive one falsehood you can derive anything. In logic there are no "small errors": one error ruins all. If 2+2 = 5 then I am the pope. In what follows, the proposition $\neg A$ will be shorthand for $A \rightarrow \perp$. In other words, "not A" will be treated the same as "if A then absurdity". The merits and disadvantages of this will be discussed in class.

$$\textbf{negation} \qquad \dfrac{\perp}{P}[\perp_E] \qquad \dfrac{\begin{array}{c} \cancel{\neg P} \\ \vdots \\ \perp \end{array}}{P}[\neg\neg_E]$$

The second rule is often known as "reduction to the absurd". It says:

> If assuming P is false gives rise to absurdity, then P must be true (with no assumptions at all).

Notice that both rules for falsehood are elimination rules. One cannot introduce falsehood.

Rules for "or": There are two *or-introduction* rules, $\vee_{I_\ell}$ and $\vee_{I_r}$ asserting that if P is true then so is $P \vee Q$, and if Q is true so is $P \vee Q$. Notice that the logician's *or* is *non-exclusive*. $P \vee Q$ means P or Q or *possibly both* are true.

or $\quad \dfrac{P}{P \vee Q}[\vee_{I_r}] \quad \dfrac{Q}{P \vee Q}[\vee_{I_\ell}] \qquad \dfrac{P \vee Q \quad \begin{matrix} \not{P} \\ \vdots \\ R \end{matrix} \quad \begin{matrix} \not{Q} \\ \vdots \\ R \end{matrix}}{R}[\vee_E]$

The elimination rule is complex. It is often called *reasoning by cases*. If you know $P \vee Q$ and you know that in either case (the P case or the Q case) you can infer R, then you know that in all cases (i.e. with no assumptions about P being true or about Q being true) you can infer R. This is also a case of *hypothetical reasoning* or a *thought experiment*. You are given that $P \vee Q$ is true but you don't know which. You "imagine" that P holds and infer R. Then you imagine that Q holds, and are still able to infer R. Then R must hold no matter what. Just from knowing $P \vee Q$ we may infer R.

1.4 Building proofs with rules

Rules are used to build **proofs**. The propositions you start with at the top are the *assumptions* or *axioms*. The conclusions of the first rules are then used as *premisses* for the next. For example, let's take another look at the inference (1) discussed in the first section of the handout

$$\frac{A \rightarrow (B \wedge C) \qquad A}{B}$$

This can now be proved using the rules of inference just given, as follows:

$$\dfrac{\dfrac{A \rightarrow (B \wedge C) \qquad A}{B \wedge C}[\rightarrow_E]}{B}[\wedge_{E_\ell}]$$

which could be described in English as follows.

> *We are given that*
>
> - *If A then $B \wedge C$.*
> - *A holds.*
>
> *Using the rule called $\rightarrow_E$ we conclude that $B \wedge C$ holds. Then using the rule $\wedge_{E_\ell}$ we conclude B holds.*

2 Problems

Problem 1 *Consider Smullyan's riddle of the three caskets each with different inscriptions:*

GOLD	SILVER	LEAD
"the portrait is not in the silver casket"	*"the portrait is not in this casket"*	*"the portrait is in this casket"*

If you know that

1. *at least one of the statements is true and at least one is false, and*
2. *the portrait is in one of the caskets,*

then which one is it in? Can you prove your claim? Write the statements $\sigma_G, \sigma_S, \sigma_L$ *on the three boxes and the conditions 1 and 2 above in symbolic form. (After we've finished covering natural deduction) try to cast your argument in natural deduction form.*

Problem 2 *Show that from the assumption* $(A \to B) \wedge (A \to C)$ *you can conclude* $A \to (B \wedge C)$ *Try both natural deduction and English language proofs.*

Problem 3 *Argue, in English, for or against the statement:*

From $(A \vee B) \wedge (\neg A)$ *infer that* B *is true.*