Logic and Computation

CLASS NOTES: HANDOUT # 2
Deductions, Parsing and Truth

September 9, 2003

1 Propositional Logic II

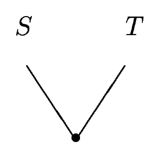
In this chapter we will spell out with a fair amount of precision what the syntatic components of propositions are, and what proofs are. We will also give precise definitions of such useful objects as as trees, and parse trees. At the end we give an alternative way to judge the validity of propositions: truth tables.

Definition 1.1 *A binary tree is any figure made up of* nodes *or* dots *"• " and* links *or* edges ——————— *containing a* **root node** *and* **leaves***, built according to the following rules.*

1. *A single node is a tree:*

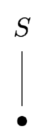
   The sole node is the root node. It is also the only leaf.

2. *If S and T are trees then so is*



   The leaves of this new tree are all those of S and of T. The root node is the new node added at the bottom.

3. *If S is a tree then so is*



   whose leaves are precisely those of S and whose root node is the new point added at the bottom.

4. *Nothing else is a tree.*

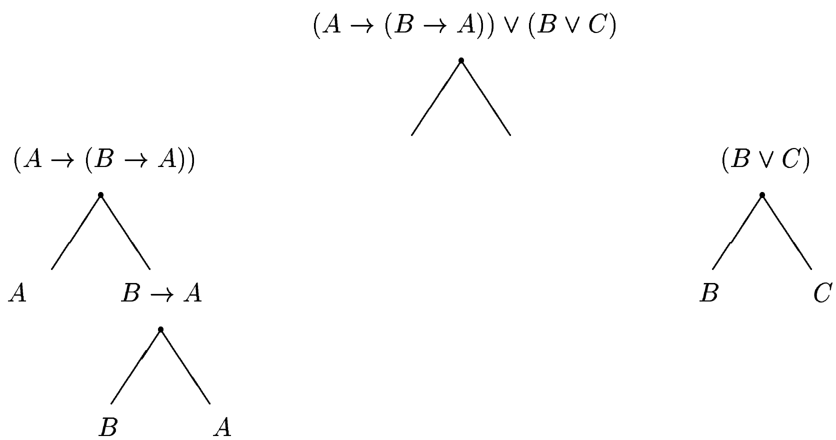
1.1 Propositions and Parse Trees

Not everything you write down using propositional letters and logical connectives is a proposition. For example $A \wedge \wedge A$ is obviously not a proposition (a good thing too, since it clearly has no meaning!). How can we be sure, in general that a given proposition is or is not well-formed? Clearly the expression just mentioned could not have been built up from atomic letters (i.e. A) using the rules that define propositions. One can see this at a glance, but in the case of bigger propositions (say ones that fill two hundred pages of a book) we need a more systematic way of checking that a proposition is well-formed.

First notice that any non-atomic proposition γ *must* be of the form

$$(\alpha \vee \beta) \qquad (\alpha \wedge \beta) \qquad (\alpha \to \beta) \qquad (\neg \alpha).$$

In each case, the appropriate logical connective $\vee, \to$, etc is called the *principal* connective of γ, and the constituent pieces α and β are called the *immediate subformulas* of γ. The only way to see if γ is really a well-formed proposition is to "tear it apart" check that it is in one of the above forms, and then check that its immediate subformulas are properly formed propositions. A nice way to visualize the decomposition of a formula into subformulas is via a *parse tree*, a labelled, binary tree (here shown growing downwards) and which is perhaps best illustrated by example: Consider $(A \to (B \to A)) \vee (B \vee C)$. The principal connective is $\vee$. The immediate subformulas are $(A \to (B \to A))$ and $(B \vee C)$. Now we take these in turn. $(A \to (B \to A))$ has $\to$ as a principal connective, and A, $B \to A$ as immediate subformulas. We can display this all at once as follows:



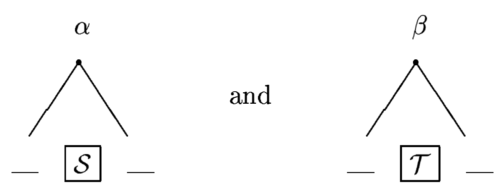
As with simple trees, we can give an "official" definition of parse trees by induction. Two clauses of the definition might read:

1. If A is a propositional letter, then

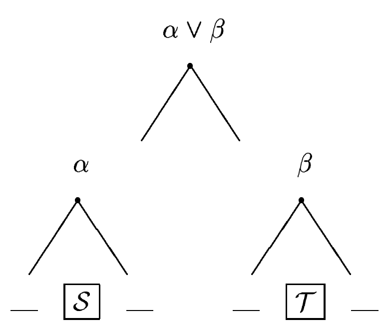
   $$A$$

   is the parse tree for A.

2. If α and β are propositions with parse trees

   then the parse tree for $\alpha \vee \beta$ is

If we use parentheses in formulas or assign a fundamental order of operations to the connectives (as with arithmetic + is before ×) then every proposition has a unique parse tree. If an expression is not a correctly formed proposition, then when we tear it apart, i.e. *parse* it, we will not be able to finish: at some point we will be left with indivisible pieces that aren't atoms.

1.2 Deduction

A Proof of a proposition is built up by putting together smaller proofs using rules. Just as with numbers, trees and propositions, we are forced to make an inductive definition, to define what a proof is in terms of smaller proofs. Proofs in natural deduction are best thought of as *labelled, ternary* trees: the nodes, instead of being unmarked dots, are now labelled with expressions (formulas, or rules), and as many as three branches may be allowed at a point. But things are a little more complicated, since as we build a proof, we are allowed to cross out or cancel certain leaves. We now supply an inductive definition. Along with the proof itself, we will need to define what the premisses and conclusions are.

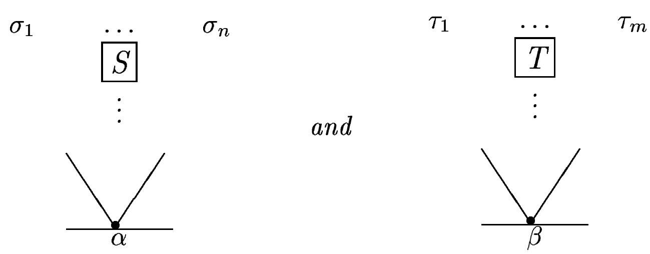
Definition 1.2 *Proofs*

1. *If α is a proposition then*

$$\alpha$$

is a proof with conclusion α and with premisses α.

2. *If S and T are proofs*

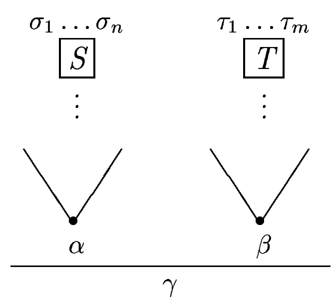


(1)

with premisses $\sigma_1, \ldots, \sigma_n$ and $\tau_1, \ldots, \tau_n$ and conclusions α and β respectively, and if

$$\frac{\alpha \qquad \beta}{\gamma}$$

is a rule with two premisses none of them cancelled, *(that is to say, if it is $\wedge_I$ or $\rightarrow_E$) then*

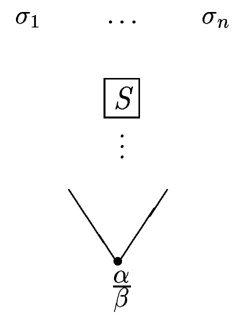


is a proof with premisses $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_m$ and with conclusion γ.

3. *If S is a proof as in (1), with premisses $\sigma_1, \ldots, \sigma_n$ and conclusion α and if*

$$\frac{\alpha}{\beta}$$

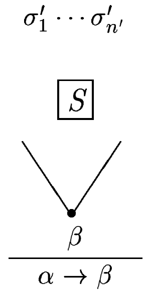
is a 1-premiss proof rule with no cancellation (i.e. $(\vee_I)$ *or* $(\wedge_E)$ *or* $(\bot)$*) then*



(2)

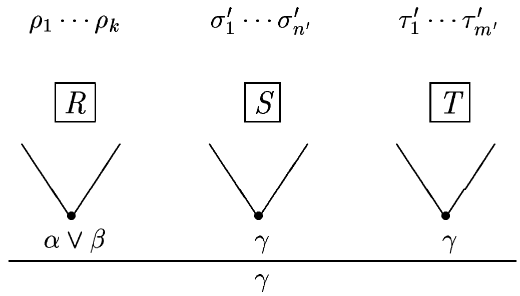
is a proof with the same premisses $\sigma_1, \ldots, \sigma_n$ *and conclusion* β*.*

4. *If* S *is a proof with premisses* $\sigma_1, \ldots, \sigma_n$ *and conclusion* β*, then*



where the premisses are what remains of $\sigma_1, \ldots, \sigma_n$ *after all occurrences of* α *have been cancelled is a proof with conclusion* $\alpha \to \beta$*.*

5. *If* R *is a proof with premisses* $\rho_1 \ldots, \rho_k$ *and conclusion* $\alpha \vee \beta$ *and if* S *and* T *are proofs with premisses* $\sigma_1, \ldots, \sigma_n$ *and* $\tau_1, \ldots, \tau_m$ *respectively and* **both** *with conclusion* γ*, then*



is a proof with conclusion γ *and with premisses all the original* $\rho_1 \cdots \rho_k$ *and the* σ*s and* τ*s remaining after all occurrences of* α *have been cancelled from the original* $\sigma_1, \ldots, \sigma_n$ *and all occurrences of* β *have been cancelled from the original* $\tau_1, \ldots, \tau_m$*.*

We have left out the description of the case of proofs whose last rule is $(\neg\neg)_E$. This is left as an exercise to the reader.

1.3 Operations and Truth Tables

1.3.1 Connectives as logical operations

Here we present a new approach to evaluating the truth of propositions. Recall that every proposition α is built up from the atomic letters that constitute its basic building blocks. [1] Some examples of propositions

[1] We sometimes indicate what those letters are by writing the proposition as $\alpha(A, B, C)$ if, for example, the letters appearing in α are A, B, and C.

using the letters A, B, C might be

$$A \wedge (B \vee C) \qquad (A \to (B \to A)) \vee (B \vee C).$$

Now ask yourself: is the first proposition above true? Well, this doesn't seem to make sense. $A \wedge (B \vee C)$ might be true or false depending on whether or not the pieces (the letters are). We need to define a way of *evaluating the truth of a proposition in terms of its constituent components (subformulas).* This is what comes next.

One way of spelling out what the word "and" means is to say

The proposition "A and B" is true if and only if the propositions A and B are both true

What about "or"? It would seem that "A or B" is true provided either A is true, or B is (or for that matter both of them). We can formally present this definition of truth of the logical connectives in a series of tables, below. We use the symbol $\top$ for "true" and "$\perp$" for "false". The symbols $\top$ and $\perp$ are called *truth values* . The following is called a table of operation for $\wedge$.and:

$\wedge$	$\top$	$\perp$
$\top$	$\top$	$\perp$
$\perp$	$\perp$	$\perp$

(3)

One can read this as follows: the truth value of "True and True" is "true". The value of "True and False" is "False", meaning $A \wedge B$ is false if A is true but B is false. The value of "False and True" is "False". And so on. This table makes the symbol "$\wedge$" into an *operation* on truth values, somethat the way "*times*" is an operation on numbers. In fact, if we let 1 stand for $\top$ and 0 for $\perp$ the table for $\wedge$ looks exactly like the multiplication table for the numbers 1 and 2 (check this!).

1.3.2 Truth tables

Another way of conveying the same information about $\wedge$ is via *truth tables.*

A	B	$A \wedge B$
$\top$	$\top$	$\top$
$\top$	$\perp$	$\perp$
$\perp$	$\top$	$\perp$
$\perp$	$\perp$	$\perp$

(4)

In a truth table for the formulas $A * B$ (where $*$ is a connective) or $\neg A$, one creates as many columns as there are variables (atomic letters) plus one more for the compound formula. We then have as many rows as there are combinations of different truth values for the atomic letters, and write the corresponding truth value for the compound formula in the last column.

We now define the truth valuations for the remaining connectives, using both tables of operations and truth tables.

or:
As an operation:

$\vee$	$\top$	$\perp$
$\top$	$\top$	$\top$
$\perp$	$\top$	$\perp$

(5)

In other words $A \vee B$ is true in all cases except when *both* A and B are false. In truth table form:

A	B	$A \vee B$
$\top$	$\top$	$\top$
$\top$	$\perp$	$\top$
$\perp$	$\top$	$\top$
$\perp$	$\perp$	$\perp$

(6)

implication:
As an operation:

$\rightarrow$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$

(7)

In truth table form:

A	B	$A \rightarrow B$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\top$

(8)

Note that we are defining $A \rightarrow B$ to be true except in the case where the antecedent A is true and the consequent B is false. It certainly seems sensible to declare the sentence "If it is a bird then it flies" to be false because we can make the antecedent true and the consequent false (consider the ostrich). Even in day-to-day discussions, the way to refute any implication is to show that the antecedent may hold while the consequent fails. How do you refute the claim, say, that zoning laws will solve a certain problem? By citing a case where zoning laws were passed, but this kind of problem wasn't solved.

But the reader may be disturbed by the fact that we let *any implication be true* whose antecedent is false. This means

If 2+2 = 5 then I'm Santa Claus.

Is considered true in propositional logic. Many would object to this in everyday discourse, but it is consistent with the logical view adopted by most mathematicians. So we'll go along with this too.[2]

not:
Since negation is a *unary* connective, that is to say, it operates on a single proposition, unlike the others (the *binary* connectives $\wedge, \vee, \rightarrow$) which operate on pairs of propositions, the tables are smaller for $\neg$.
As an operation:

$\neg$	
$\top$	$\bot$
$\bot$	$\top$

(9)

In truth table form:

A	$\neg A$
$\top$	$\bot$
$\bot$	$\top$

(10)

Truth Tables for Compound Formulas

We can now extend truth tables to any proposition γ, simply by writing a column for every subformula of γ starting with its atomic subformulas (i.e. the atomic letters occurring in it). We then fill in all possible combinations of values $\top$ and $\bot$ for the letters and proceed to compute the truth values using the basic tables for the connectives given above. We illustrate for the proposition:

$$(A \rightarrow (B \rightarrow A)) \rightarrow (B \vee C).$$

We need to identify the subformulas in this formula. They are:

$$(A \rightarrow (B \rightarrow A)), (B \vee C), B \rightarrow A, A, B, C.$$

[2] Proponents of a form of logic called *minimal logic* reject this idea, as do various other philosophical schools.

There are three atomic subformulas: A, B, C and hence we will need 8 columns to write out all possible assignments of truth values to them. We start by filling in just this part.

A	B	C	$B \to A$	etc...
$\top$	$\top$	$\top$		
$\top$	$\top$	$\bot$		
$\top$	$\bot$	$\top$		
$\top$	$\bot$	$\bot$		
$\bot$	$\top$	$\top$		
$\bot$	$\top$	$\bot$		
$\bot$	$\bot$	$\top$		
$\bot$	$\bot$	$\bot$		

Now the rest of the truth values are determined by the operational values of the connectives, i.e. by the basic tables above. For example we fill in $\top$ for $B \to A$ in every row except those where B is true and A is false. We fill in one more column to illustrate:

A	B	C	$B \to A$	etc...
$\top$	$\top$	$\top$	$\top$	
$\top$	$\top$	$\bot$	$\top$	
$\top$	$\bot$	$\top$	$\top$	
$\top$	$\bot$	$\bot$	$\top$	
$\bot$	$\top$	$\top$	$\bot$	
$\bot$	$\top$	$\bot$	$\bot$	
$\bot$	$\bot$	$\top$	$\top$	
$\bot$	$\bot$	$\bot$	$\top$	

Now we fill in the rest:

A	B	C	$B \to A$	$A \to (B \to A)$	$B \vee C$	$(A \to (B \to A)) \to (B \vee C)$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$

Thus the proposition $A \to (B \to A) \to (B \vee C)$ is true in all cases except when B and C are all false. Notice, by the way, that the proposition $A \to (B \to A)$ is always true. There is no assignment of truth values to its atomic subformulas which can falsify it. Does this make sense? Does this make it a special sort of proposition?

1.4 Tautologies and Contradictions

Definition 1.3 *A proposition α is called a* **tautology** *if its column contains all $\top$s in the truth table for α. It is called a* **contradiction** *if its column contains only $\bot$s.*

For example, we have already seen that $A \to (B \to A)$ is a tautology. In english, we might render this as

"if A holds then anything implies A" .

Could you argue the validity of such a claim in English? Is there a natural deduction proof of it?

You may recall from the first page of the Week #1 handout that we studied several inferences whose validity didn't really seem to depend on the content of the component statements, such as:

If Bill can run for office then Bill is over 35 and (Bill is) a US citizen.
Bill can run for office

Bill is over 35

We ended up replacing such statements as "Bill is over 35" by letters $A, B, C \ldots$. This just helps to underscore the fact that we are studying inferences that are true by virtue of their *form*. Consider

$$\frac{\frac{\cancel{A \wedge B}}{A}}{(A \wedge B) \to A}.$$

It really doesn't matter whether A or B are true. We are really asserting that "A follows from $A \wedge B$". Checking that proposition $(A \wedge B) \to A$ is a tautology via truth tables (check it!) seems to be just another way to say this. But is it really? The deep question here is:

Are the tautologies precisely the same propositions as the ones provable using Natural Deduction?

A related question can be asked about the following two definitions, which will prove useful.

Definition 1.4 *We say two propositions α and β are* **provably equivalent** *if both $\alpha \to \beta$ and $\beta \to \alpha$ are provable using natural deduction. We say they are* **logically equivalent** *or* **truth-table equivalent** *if both $\alpha \to \beta$ and $\beta \to \alpha$ are tautologies.*

The related question is: *are provable and logical equivalence the same thing ?* More on this later. Now, a few exercises.

2 exercises

DUE OCT 17

Problem 1 *Draw a parse tree for the following formula*

$$A \to (\neg(B \wedge C) \to (C \to (D \to A)))$$

Problem 2 *Which of the following are tautologies?*

1. $(\alpha \vee \alpha) \to \alpha$.
2. $((\alpha \vee \beta) \vee \gamma) \longleftrightarrow (\alpha \vee (\beta \vee \gamma))$.
3. $(\alpha \to \beta) \longleftrightarrow \neg(\neg\alpha \wedge \beta)$.
4. $(\alpha \wedge (\alpha \vee \beta)) \longleftrightarrow \alpha$
5. $\neg(\alpha \vee \beta) \longleftrightarrow (\neg\alpha \wedge \neg\beta)$.
6. $\alpha \to (\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.
7. $(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$.
8. $\alpha \to \neg\neg\alpha$
9. $(\alpha \to \beta) \vee (\beta \to \alpha)$.
10. $((\alpha \to \beta) \to \alpha) \to \alpha$.
11. $(\alpha \to \beta) \longleftrightarrow (\neg\alpha \vee \beta)$.
12. $\alpha \vee \neg\alpha$
13. $\neg\neg\alpha \longleftrightarrow \alpha$

Problem 3 *Prove the tautologies in the preceding list by natural deduction. To prove something of the form* $\alpha \longleftrightarrow \beta$ *you must give a proof of* $\alpha \rightarrow \beta$ *and another proof of* $\beta \rightarrow \alpha$.

Problem 4 *Give an argument in English to convince someone of the truth of*

1. $\neg(A \vee B) \longleftrightarrow (\neg A \wedge \neg B)$
2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
3. $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$

Does looking at the truth tables for these formulas help? What about natural deduction proofs?

Problem 5 *One often uses the connective* $\longleftrightarrow$ *(bi-implication), say, in*

$$A \longleftrightarrow B$$

as shorthand for $(A \rightarrow B) \wedge (B \rightarrow A)$. *To check if* $A \longleftrightarrow B$ *is true one must check that* both $A \rightarrow B$ *and* $B \rightarrow A$ *are true. But we could consider it as a new connective. Then what should its truth table be? Fill in the last column.*

A	B	$A \longleftrightarrow B$
$\top$	$\top$	
$\top$	$\bot$	
$\bot$	$\top$	
$\bot$	$\bot$	

True or false: the statement "α and β are logically equivalent" is the same as saying that "$\alpha \longleftrightarrow \beta$ is a tautology."

Problem 6 *Finish the inductive definition of proof, by supplying the definition for the case where the last rule used was* $(\neg\neg_E)$.

Problem 7 *A* **symmetric** *binary tree is one whose "right half" is the mirror image of its left half. Make this definition precise by giving an inductive definition of a symmetric binary tree.*

Problem 8

(i) *Prove that* α *is a tautology if and only iff* $\neg(\alpha)$ *is a contradiction.*

(ii) *Prove or disprove: "$\alpha \vee \beta$" is a tautology if and only if either α is or β is."*

Problem 9 *Justify the following new rule of inference (here called "3-or") by showing that it is superfluous, i.e. that the same conclusion can already be obtained using the NK-calculus (Gentzen's natural deduction).*

$$\frac{((\alpha \vee \beta) \vee \gamma) \quad \alpha \rightarrow \theta \quad \beta \rightarrow \theta \quad \gamma \rightarrow \theta}{\theta}\,\text{3-}\vee$$

In other words, give a derivation of θ *using the same premisses and (as many rules as you want in) the NK-calculus.*