Math 131

CLASS NOTES
(Supplement)
Predicate Logic

1 Introduction

We have studied propositional logic, the logic of statements made up of basic or atomic true-or-false propositions using $\neg, \vee, \wedge, \rightarrow$. This logic is not sufficiently expressive to capture all mathematical reasoning however. We have no way, in propositional logic, of talking about *individuals*, members of a set which may have or fail to have certain properties, and there is no way of *quantifying* over individuals, i.e. talking about *some* or *all* individuals in a set. For example consider the following statement about the natural numbers, (which asserts that there is no biggest number).

For every number x there is a number y such that $x < y$.

which is written symbolically:

$$\forall x \exists y (x < y)$$

where $\forall$ ("for all") is called a universal quantifier, and $\exists$ ("exists") is called an existential one. Notice that with propositional logic we cannot even make sense of the subformula

$$x < y$$

This is an atomic formula, with no propositional connectives, yet it is neither true nor false until we assign values to x or y. By using quantifiers $\exists, \forall$ we can bind these variables, and construct a statement which is capable of being true or false, such as $\forall x \forall y (x + y = y + x)$. We now develop the syntax and semantics of the calculus of predicates, or predicate logic, in which such concepts can be formalized, and extend the notions of proof and validity to include these new notions.

2 Languages, terms and formulas

Assume given a countably infinite set of variables. We recall that a *language* $\mathcal{L}$ is a triple $\langle \mathcal{C}, \mathcal{F}, \mathcal{R} \rangle$ where $\mathcal{C}$ is a set of *constants*, $\mathcal{F}$ a set of *function symbols* f^n paired with a natural number n called the *arity* of the function symbol f, and $\mathcal{R}$ a set of *relation symbols* R^n of arity n.

When the context makes it clear, we will often display a language by directly listing its function, relation and constant symbols, without bothering to name the sets $\mathcal{C}, \mathcal{F}, \mathcal{R}$. For example the language of arithmetic with ineqaulity $\mathcal{L}_{A,\leq} = \langle 0, 1, +^2, \times^2, \leq^2 \rangle$ has constant symbols $0, 1$, function symbols $+, \times$ each of arity 2, and relation symbol $\leq$ of arity 2.

In propositional logic we never made use of relation symbols. We will now use them to build atomic formulas. But first we recap the definition of terms, here generalized to include expressions with variables.

Definition 2.1 *Let $\mathcal{L}$ be a language. Then the set of* **open terms** *over $\mathcal{L}$ is given inductively by the following rules.*

1. *If c is a constant in $\mathcal{L}$ then c is a term.*
2. *Any variable x is a term.*

3. *If f is a function symbol in $\mathcal{L}$ of arity n and $t_1, \ldots, t_n$ are terms, then $ft_1, \ldots, t_n$ is a term.*

4. *Nothing else is a term.*

A term is said to be **closed** *(or* **ground***) if it contains no free variables. if t_1 and t_2 are terms and x is a variable, we use the notation $t_1[t_2/x]$ to denote the term resulting from the replacement of every occurrence of x in t_1 by t_2.*

Problem 1 *Give a rigorous definition of $t_1[t_2/x]$ by induction on the structure of t_1. [For example, the atomic case is $c[t_2/x] := c$, the result of substituting t_2 for x in c is c, since c is a constant, hence a term in which x does not even occur.]*

We are now ready to define *formulas*: the assertions in predicate logic.

Definition 2.2 *Let $\mathcal{L}$ be a language. Then the set of* **open formulas** *over $\mathcal{L}$ is given inductively by the following rules.*

1. *if R is a relation symbol of arity n and $t_1, \ldots, t_n$ terms, then $R(t_1, \ldots, t_n)$ is a formula (called an* **atomic** *formula).*

2. *If α and β are formulas then so are*

$$(\alpha \vee \beta) \quad (\alpha \wedge \beta) \quad (\alpha \to \beta) \quad (\neg \alpha)$$

3. *If α is a formula and x is a variable then $\exists x \alpha$ and $\forall x \alpha$ are formulas. α is said to be in the* **scope** *of the quantifier $\forall x$ or $\exists x$.*

4. *Nothing else is a formula.*

Any occurrence of the variable x in a formula not in the scope of a a quantifier is said to be a **free** occurrence. Otherwise it is a **bound** occurrence. Thus, if x is free in α it is bound in $\exists x \alpha$ and $\forall x \alpha$. A formula with no free variables is called **closed**.

Problem 2 *Let $\mathcal{L}$ be the language $\langle b, \ell^2 \rangle$ where b is the constant whose intended interpretation is "my baby" and $\ell(x, y)$ will express "x loves y". Transcribe the following sentences into formulas over $\mathcal{L}$.*

1. *Everybody loves my baby.*

2. *My baby don't love nobody but me.*

Cook up a language to capture Lewis Carroll's interesting assertions:

1. *Nobody who can handle crocodiles is despised.*

2. *Babies cannot handle crocodiles.*

We will discuss some of the consequences in later lectures. Note than we can give an inductive definition of free and bound occurrence. If x occurs in any of the terms t_i then that occurrence of x is free in $R(t_1, \ldots, t_n)$. A free occurrence of x in α is also a free occurrence of x in $\neg\alpha$, $\alpha \vee \beta$, etc.

Problem 3 *Complete the inductive definition of free and bound occurrence. Does x occur free in $P(x) \wedge \forall x \beta(x)$? Does it occur bound?*

If α is a formula, we also define the substitution $\alpha[t/x]$ to denote the formula obtained by replacing every *free* occurrence of x in α by t.

Problem 4 *Perform the indicated substitutions.*

1. $(P(x, y) \to \forall x Q(x, y))[(u + 2)/x]$

2. $(P(x,y) \to \forall x Q(x,y))[(u+2)/y]$

3. $(P(x,y) \to \forall x Q(x,y))[(u+2)/z]$

Give a rigorous definition of substitution $\alpha[t/x]$ by induction on the structure of the formula α.

The definition of subformula and immediate subformula is just like for propositional logic, except that now we have to deal with quantified formulas. α is the immediate subformula of $\exists x\alpha$ and $\forall x\alpha$. A subformula of a quantified formula $Qx\alpha$ (where Q is $\exists$ or $\forall$) is either the whole formula, or α or a subformula of α. The only subformula of an atomic formula is itself.

Problem 5 *List the subformulas of $(P(x,y) \to \forall x Q(x,y))$. Give an inductive defintion of a parse tree for a formula of predicate logic. [Note that the leaves of a parse tree must be atomic formulas, which are no longer just letters.]*

3 Proofs in predicate logic

In order to extend natural deduction to predicate logic we need to add only four more rules:

Intro	**Elim**
$\dfrac{\begin{matrix}\vdots\\ P(x)\end{matrix}}{\forall x P}\,(\forall_I\ \dagger)$	$\dfrac{\forall x P}{P(t)}\,(\forall_E)$
$\dfrac{P(t)}{\exists x P}\,(\exists_I)$	$\dfrac{\exists x P \qquad \begin{matrix}\cancel{P(x)}\\ \vdots\\ R\end{matrix}}{R}\,(\exists_E\ \ddagger)$

($\dagger$) x is not free in any premiss in the proof of $P(x)$ ($\ddagger$) x is not free in R or in any premiss used in its proof *except* $P(x)$.

Note the very important *side conditions* given as footnotes. We now discuss these in turn.

$\forall$-intro The key use of *variables* in mathematical reasoning is to prove properties that are true *in general*, for all individuals rather than a specific one. For example consider the following theorem and its proof.

Theorem 3.1 *There are infinitely many primes: for every prime x there is a prime y with $y > x$*

proof: Let x be prime. Let $x_1, \ldots, x_n$ be the set of primes smaller than x. Let $z = x \cdot x_1 \cdots (x_n + 1)$. Claim z is not divisible by any of the x_i or x, so any prime y dividing x must be greater than x. (the claim is easy...if x_i divides z it must also divide 1, a contradiction.) ▮

Now observe that *we never said what number x was.*. It might have been 7 or 13 or 29, and the above argument would show that a greater prime existed. By arguing about a *letter x about which no special assumptions were made* we proved a theorem about all numbers. This is captured by the $\forall$-intro rule

$$\frac{\begin{array}{c}\vdots \\ P(x)\end{array}}{\forall x P(x)}$$

which allows you to conclude $\forall x P(x)$ just from having shown it for x *provided you never used x in your assumptions*. To be even more precise about how this rule is used, we extend our inductive definition of a natural deduction proof in the same style as in the preceding handout.

Suppose S is a proof with premisses $\sigma_1, \dots, \sigma_n$ and conclusion P and that x *does not occur free in any of the σ_i*. Then

$$\begin{array}{c} \sigma_1 \qquad \dots \qquad \sigma_n \\ \\ \boxed{\text{S}} \\ \vdots \\ \\ \dfrac{P}{\forall x P(x)}\,\forall_I \end{array} \tag{1}$$

is a proof with the same premisses $\sigma_1, \dots, \sigma_n$ and conclusion $\forall x P$.

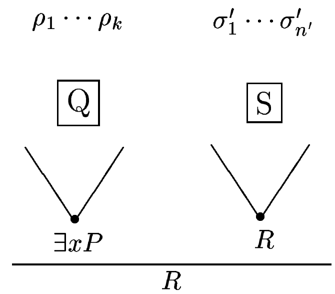
$\forall$-elim: The elimination rule for $\forall$ is more obvious. If you have already established $\forall x P$ it's legal to substitute in for x any term in the language.

$\exists$-elim One should look at the exist-elimination rule as an analogue to $\vee$-elim.

$$\frac{\exists x P \qquad \begin{array}{c}\cancel{P(x)} \\ \vdots \\ R\end{array}}{R}\,(\exists_E)$$

Suppose someone tells you that for some x the property $P(x)$ holds and you are able to show that *no matter what x you pick*, from $P(x)$ you can infer the conclusion R. Then you are able to conclude R just from $\exists x P(x)$. The only way to be sure that the argument really doesn't depend on the choice of x is to enforce the same kind of condition we saw in $\forall$-intro: x must not occur free in any premiss used to prove R *except* $P(x)$, and x must not occur free in R. In the same spirit as (1) above, we now extend the formal inductive definition of proof to include this rule.

If Q is a proof with premisses $\rho_1 \dots, \rho_k$ and conclusion $\exists x P$ and if S is a proof with premisses $\sigma_1, \dots, \sigma_n$ **in which x doesn't occur free** in any σ_i **except** $P(x)$ and with a conclusion R in which x doesn't occur free, then



(2)

is a proof with conclusion R and with premisses all the original $\rho_1 \cdots \rho_k$ and the σs remaining after all occurrences of $P(x)$ have been cancelled from the original $\sigma_1, \ldots, \sigma_n$.

Problem 6 *Extend the inductive definition of natural deduction proof in the style of (1) and (2) to include the rules $\exists$-intro and $\forall$-elim.*

With these rules we can now justify such inferences as the Aristotelian syllogism:

$$\frac{\text{All men are mortal.} \quad \text{Socrates is a man.}}{\text{Socrates is mortal.}}$$

3.1 Exercises

Problem 7 *Prove the following using natural deduction.*

1. $\forall x(\varphi(x) \to \varphi(x))$.
2. $\forall x \varphi(x) \to \exists x \varphi(x)$.
3. $\forall x(\varphi(x) \wedge \psi(x)) \to (\forall x \varphi(x)) \wedge (\forall x \psi(x))$.
4. $\forall x(\varphi \to \psi(x)) \to (\varphi \to \forall x \psi(x))$, *where x is not free in $\varphi(x)$.*