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Construction of Rational Elliptic Surfaces with Mordell-Weil Rank 4

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Introduction

In this master thesis we provide a geometric construction of rational elliptic surfaces of Mordell-Weil rank four. Most of the techniques we use are similar to those in [**Sal09**] and [**Fus06**] for the construction of rational elliptic surfaces with higher Mordell-Weil rank.

Elliptic Curves. Elliptic curves are an important object of study in algebraic geometry, number theory, cryptography, as well as in many other scientific subjects. In this thesis we always deal with elliptic curves defined over an algebraically closed field k of characteristic 0 or over a function field over k with finite transcendence degree over k. An elliptic curve is a pair (E, O), where E is a curve of genus 1 and O is a point on E such that E has a group structure with O as the zero element. The simplest way to think of an elliptic curve is in the Weierstrass form: we can define an elliptic curve as the set of solutions in the projective plane over a given field k of the equation

$$y^2 = x^3 + Ax + B,$$

where A and B are two parameters in k such that $4 \cdot A^3 + 27 \cdot B^2 \neq 0$. In this shape every elliptic curve has a group structure given by the geometric rule "three collinear points add up to zero", and the zero element is given by the point at infinity.

Even if it is true that all the elliptic curves can be seen in this form, we will not use this representation, since our approach needs a more general point of view that fits some different requests.

Rational Elliptic Surfaces: one Object, two Points of View. We work over an algebraically closed field k with characteristic zero.
An elliptic surface over k is an algebraic surface $\mathscr{E}$ over k, equipped with a flat morphism $\pi : \mathscr{E} \to B$, where B is a projective curve and the following requirements are satisfied:

- the morphism π is an elliptic fibration: $\pi^{-1}(t)$ is a curve of genus 1, for almost all $t \in B(k)$;
- there is a zero section, that is a morphism $\sigma_0 : B \to \mathscr{E}$ such that $\pi \circ \sigma_0 = \mathrm{id}_B$.

Moreover, we suppose that there is at least a $t \in B(k)$ such that $\pi^{-1}(t)$ is singular.

Elliptic surfaces constitute an important class of algebraic surfaces, since they can be seen as elliptic curves over a function field or as families of elliptic curves over the ground field. This two folded description makes these objects interesting and simpler to study.

We will deal just with a subclass of elliptic surfaces, focusing our attention on the rational ones, i.e. elliptic surfaces that are birational to the projective plane. This restriction implies that the curve B is the projective line. Thus, in terms of the above description, a rational elliptic surface over k can be seen as an elliptic surface over the function field $k(t) \cong k(\mathbb{P}^1)$ or as a linear pencil of plane cubic curves.

From the former point of view we use the theory of elliptic curves over function fields to describe the invariants of an elliptic surface; the latter gives a natural geometric construction. Consider a smooth cubic F and a different cubic G. The map

$$\begin{aligned} \pi : \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1 \\ (x, y, z) &\longmapsto (F(x, y, z), G(x, y, z)) \end{aligned}$$

is not well-defined at the intersection points of those curves, i.e. the base locus of the linear pencil of cubics generated by F and G. In order to obtain a morphism, we blow-up the base points of the pencil. In this way we obtain a rational surface endowed with a flat morphism such that each fiber is a genus 1 curve, i.e. a rational elliptic surface.

Every possible rational elliptic surface is isomorphic to the blow-up of $\mathbb{P}^2$ at the base points of a linear pencil of cubics, as shown in [**Mir89**].

This construction was already used in order to study rational elliptic surfaces: Shioda gave the construction of rational elliptic surfaces with rank eight in [**Shi90**], Fusi gave the construction of those with rank seven and six in [**Fus06**] and Salgado gave the construction of those with rank five in [**Sal09**]. We use the same techniques to construct rational elliptic surfaces with rank four. In order to determine which pencils of cubics induce a rational elliptic surface with given rank, we use the Shioda-Tate formula, which gives a criterion to determine the rank of an elliptic surface by the number of components of the reducible fibers (these correspond to the blow-up of singular cubics in the pencil).

Since the Néron-Severi rank is fixed (and equal to ten), the lower the rank the wider the range of possible fiber types that can occur. In our case there are six possible fiber-types for rational elliptic surfaces of rank four without torsion and one for rational elliptic surfaces of rank four with torsion.

Our construction is case-by-case: we focus on a certain fiber type that leads to a rational elliptic surface with rank four and we find a linear pencil of cubics inducing that fiber type, via its singular members.

As in the papers that studied higher rank rational elliptic surfaces, we want to go further with our construction; namely, we want to check whether the exceptional curves over the base points of the pencil generate the Mordell-Weil group of the induced surface. The tool to perform this action can be found in [**OS91**]: for every rational elliptic surfaces, its Mordell-Weil group, modulo torsion, has a lattice structure, together with a bilinear symmetric pairing $\langle , \rangle$. For any set of independent elements $\{P_1, \dots, P_r\}$ in the Mordell-Weil group of a rational elliptic surface of rank r, we can build a symmetric matrix A whose elements are given by $a_{i,j} = \langle P_i, P_j \rangle$. The determinant of this matrix measures how the considered elements $P_1, \dots, P_r$ are far to generate the Mordell-Weil lattice: the determinant of A is equal to a^2 times the determinant of the Mordell-Weil lattice, for some integer a. This integer a is exactly the index of the sublattice generated by the considered elements if it is different from zero (if $a = 0$, the chosen elements are dependent). In this thesis we always had $a = 1$, that is, we were always able to generate the full Mordell-Weil lattice. This implies that in all the non-torsion cases we were able to generate the Mordell-Weil group; in the torsion case this is true again, since the exceptional curves above the base points generate the Mordell-Weil lattice (which is a subgroup of index 2 of the Mordell-Weil group) and they also generate the torsion component of the Mordell-Weil group.

CHAPTER 1

Preliminaries

In this chapter we list a series of basic results needed for the construction of rational elliptic surfaces.

1.1. Basic Background and Notation

Let k be an algebraically closed field of characteristic zero. A **projective algebraic set** is a subset X of $\mathbb{P}^n$ such that there exists a set S of homogeneous polynomials in $n+1$ variables giving the following equality:

$$X = \{x \in \mathbb{P}^n \mid f(x) = 0 \text{ for all } f \in S\}.$$

A projective algebraic set X is said to be irreducible if X cannot be written as the disjoint union of two proper Zariski-closed subsets.

A **projective variety** is an irreducible algebraic subset of $\mathbb{P}^n$, with the induced topology. A **quasi-projective variety** is an open subset of a projective variety. The **dimension** of a projective or quasi-projective variety is its dimension as topological space.

We will mainly focus on varieties of dimension less than or equal to 2, that is: points, curves and surfaces.

Let $X \subseteq \mathbb{P}^n$ be a projective variety. An irreducible algebraic subset of X with the induced topology is called **subvariety** of X.

If $\varphi : X \to Y$ is a map, for every $U \subseteq Y$ we will denote by $\varphi^{-1}(U)$ the subset of X consisting of the elements $x \in X$ such that $\varphi(x) \in U$. If $U = \{P\}$, we will write $\varphi^{-1}(P)$ instead of $\varphi^{-1}(\{P\})$.

A map $f : X \dashrightarrow k$ is **regular at a point** $P \in X$ if there is an open neighborhood U of P in X and homogeneous polynomials $g, h \in k[x_0, \dots, x_n]$ of the same degree, such that h does not vanish on U and $f = g/h$ on U. We say that f is a **regular function** if it is regular at all $P \in X$.

Let X and Y be two varieties. Let $\varphi : X \to Y$ be a continuous function. We say that φ is a **morphism** if for every open set $V \subseteq Y$ and for every regular function $f : V \to k$, the function $f \circ \varphi : \varphi^{-1}(V) \to k$ is regular.

Let $\varphi : X \to Y$ be a morphism. If there exists a morphism $\psi : Y \to X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$, we say that φ is an **isomorphism**.

1.2. Divisors

Let X be a projective variety of dimension n. An irreducible subvariety Y of X of dimension $n-1$ is called a **prime divisor** on X. The free abelian group generated by prime divisors is called the **divisor group** of X. We will denote this group by $\mathrm{Div}(X)$. An element $D \in \mathrm{Div}(X)$ is called a **divisor** on X.

For every $D \in \mathrm{Div}(X)$ one can write

$$D = \sum_Y n_Y Y,$$

where the sum ranges over prime divisors and n_Y is an integer, which is equal to zero for almost all Y.

If $n_Y \geq 0$ for all prime divisors Y, we say that D is **effective**.

Let f be a non-zero function on X. For each prime divisor Y of X, we will denote the valuation of f at Y by $v_Y(f)$. This number is zero for almost all Y (See [Har77], page 131). The map

$$\begin{aligned} \mathrm{div} : k(X)^{\times} &\longrightarrow \mathrm{Div}(X) \\ f &\longmapsto \sum v_Y(f) Y \end{aligned}$$

is well defined and, indeed, it is a group morphism whose image defines a subgroup of $\mathrm{Div}(X)$, called the (sub)group of the **Principal Divisors** on X, denoted by $\mathrm{PDiv}(X)$.
To improve the readability, we will write (f) instead of $\mathrm{div}(f)$.

Let D and D' be two divisors. If

$$D' = D + (f), \qquad \text{for some } f \in k(X)^{\times},$$

we say that D and D' are **linearly equivalent** and we write $D \sim D'$.

The quotient $\mathrm{Div}(X)/\mathrm{PDiv}(X)$ is called the **Picard Group** of X and is denoted by $\mathrm{Pic}(X)$.

This group fits into an exact sequence

$$1 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow NS(X) \longrightarrow 0,$$

where $\mathrm{Pic}^0(X)$ is the the group of divisors which are algebraically equivalent to 0 and

$$NS(X) = \frac{\mathrm{Pic}(X)}{\mathrm{Pic}^0(X)}.$$

This is a finitely generated abelian group, called the **Néron-Severi group** of X.

1.3. Surfaces

Let X be an algebraic variety. For all open sets $U \subseteq X$, we denote by $\mathscr{O}(U)$ the ring of regular functions on U. If V is an open subset of U we can define the map $\rho_{U,V} : \mathscr{O}(U) \to \mathscr{O}(V)$ as the usual restriction. It is easy to check that $\mathscr{O}$ is indeed a sheaf, called **sheaf of regular functions** on X.
If P is a point on X, we define the **local ring of** P **on** X, $\mathscr{O}_{P,X}$ to be the ring of germs of regular functions on X near P, (i.e. the stalk of $\mathscr{O}$ at P).

The theorems below are central in intersection theory on a surface. Let X be a non-singular projective surface over an algebraically closed field k, C and D be two curves on X. If P is a point in both C and D, we say that C and D meet **transversally** at P if the local equations f of C and g of D generate the maximal ideal of P in $\mathscr{O}_{P,X}$.

Theorem 1.3.1. *Let X be a non-singular projective surface over an algebraically closed field k. There is a unique pairing*

$$\begin{aligned} \mathrm{Div}(X) \times \mathrm{Div}(X) &\longrightarrow \mathbb{Z} \\ (C, D) &\longmapsto (C \cdot D), \end{aligned}$$

such that

(1) *if C and D are non-singular curves meeting transversally, then $(C \cdot D) = \#(C \cap D)$,*
(2) *it is symmetric: $(C \cdot D) = (D \cdot C)$,*
(3) *it is additive: $((C_1 + C_2) \cdot D) = (C_1 \cdot D) + (C_2 \cdot D)$,*
(4) *it depends only on the linear equivalence classes: if $C_1 \sim C_2$ then $(C_1 \cdot D) = (C_2 \cdot D)$.*

Proof. See [Har77], page 358. □

Let X be a non-singular projective surface over an algebraically closed field k. Let C and D be two curves in X with no common irreducible component and $P \in C \cap D$. We define the **intersection multiplicity** $(C \cdot D)_P$ of C and D at P as the dimension of $\mathscr{O}_{P,X}/(f,g)$ as a k-vector space. We have the following equality:

$$(C \cdot D) = \sum_{P \in C \cap D} (C \cdot D)_P.$$

Again, see [Har77] for a complete proof.

We now consider $C \cap D$ as a scheme. The ideal sheaf defining C (resp D) is the invertible sheaf $\mathscr{O}_X(-C)$ (resp $\mathscr{O}_X(-D)$); now define

$$\mathscr{O}_{C \cap D} = \frac{\mathscr{O}_X}{\mathscr{O}_X(-C) + \mathscr{O}_X(-D)}.$$

For every $P \in C \cap D$ we have $(O_{C \cap D})_P = \mathscr{O}_P/(f,g)$. This leads us to the following equality, using also the equation above:

$$(C \cdot D) = \dim(H^0(X, \mathscr{O}_{C \cap D})).$$

Now, for every sheaf $\mathscr{F}$ on X, define the **Euler-Poincaré characteristic** of $\mathscr{F}$ as:

$$\chi(\mathscr{F}) = \sum_{i=0}^{\infty} (-1)^i \dim(H^i(X, \mathscr{F})).$$

The following theorem is crucial in intersection theory: it enables us to extend the intersection form to any two divisors on a surface, by letting us replace any of the two divisors with a linear equivalent one.

Theorem 1.3.2. *Let X be a non-singular projective surface over an algebraically closed field k. For every $\mathscr{F}, \mathscr{G} \in \mathrm{Pic}(X)$ (seen as the group of isomorphism classes of invertible sheaves on X) define*

$$(\mathscr{F} \cdot \mathscr{G}) = \chi(\mathscr{O}_X) - \chi(\mathscr{F}^{-1}) - \chi(\mathscr{G}^{-1}) + \chi(\mathscr{F}^{-1} \otimes \mathscr{G}^{-1}).$$

Then $(\ .\ )$ is a bilinear form on $\mathrm{Pic}(X)$ such that if C and D are two irreducible curves on X meeting transversally, then

$$(\mathscr{O}_X(C) \cdot \mathscr{O}_X(D)) = (C \cdot D).$$

Proof. See [Bea96], page 4. □

This theorem allows us to extend the previous definition of intersection between transversal divisors to all divisors. We can write $(C \cdot D)$ in place of $(\mathscr{O}_X(C) \cdot \mathscr{O}_X(D))$, since those two quantities are equal where $(C \cdot D)$ is defined.

Corollary 1.3.3. *Let X be a non-singular projective surface over an algebraically closed field k. Let C be a smooth curve over k. Let $f : X \to C$ be a surjective morphism. Then for every fiber $F = f^{-1}(P)$ of f we have $F^2 = 0$.*

Proof. See [Bea96], page 4. □

A divisor D on a surface X is **numerically equivalent to zero** if

$$(D \cdot E) = 0, \qquad \text{for all divisors } E.$$

In this case we write $D \equiv 0$. We say that D and E are **numerically equivalent** if $D - E \equiv 0$.

1.4. Blowing Up

We will follow Hartshorne's construction (see [Har77]). We will first define the blow-up of $\mathbb{A}^n$ at the origin O. Consider the quasi-projective variety given by the product $\mathbb{A}^n \times \mathbb{P}^{n-1}$. We will denote by $x_1, \ldots, x_n$ the affine coordinates of $\mathbb{A}^n$ and by $y_1, \ldots, y_n$ the homogeneous coordinates of $\mathbb{P}^{n-1}$.

The **blow-up** of $\mathbb{A}^n$ at O is the closed subset X of $\mathbb{A}^n \times \mathbb{P}^{n-1}$, given by the equations:

$$\begin{cases} x_1y_2 & = & x_2y_1 \\ x_1y_3 & = & x_3y_1 \\ \vdots & = & \vdots \\ x_iy_j & = & x_jy_i \\ \vdots & = & \vdots \end{cases}$$

1.4.1. Properties.

(1) The projection $\mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$ induces a natural morphism $\varphi : X \to \mathbb{A}^n$.
(2) For every point $P \in \mathbb{A}^n$ there is a unique element in $\varphi^{-1}(P)$, except for $P = O$. Indeed, φ induces an isomorphism

$$\varphi : X \setminus \varphi^{-1}(O) \longrightarrow \mathbb{A}^n \setminus O.$$

(3) $\varphi^{-1}(O) \cong \mathbb{P}^{n-1}$

Now, we can define the blow-up for every closed subvariety Y of $\mathbb{A}^n$ at $P \in Y$. First of all, we can assume that $P = O$: if this is not the case, we can translate P to the origin. The **blow-up** of Y at O is defined as

$$\tilde{Y} = \varphi^{-1}(Y \setminus O).$$

In the case $n = 2$, the curve $E = \varphi^{-1}(O)$ is called the **exceptional curve** above the origin. For each curve C in $\mathbb{A}^2$ passing through the origin, we define two other curves: the total inverse image of C is called **proper transform** of C and consists of E and another curve C', called the **strict transform** of C. All the other curves in $\mathbb{A}^2$ are isomorphic to their pre-image under the blow-up.

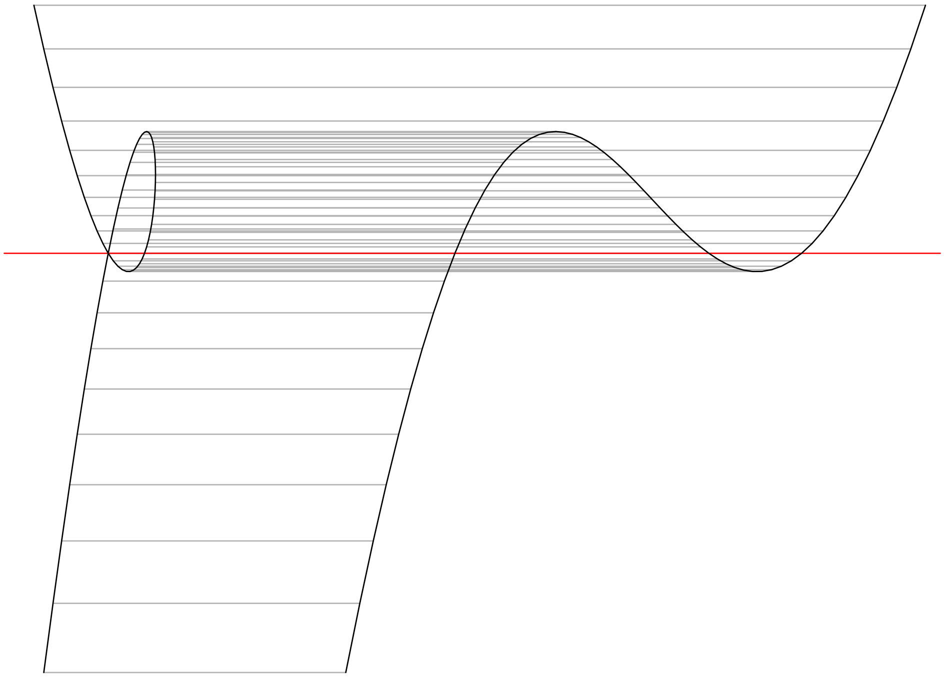


FIGURE 1.1. A node $Y^2 = X^3 + X^2$ on the left and its blow-up at $(0,0)$ on the right. The red line is the exceptional curve above the origin.

1.5. Elliptic Curves

1.5.1. Assumptions. An **elliptic curve** over the field k is a pair (E, O), where E is a curve of genus 1 defined over k and O is a point on $E(k)$. We generally omit the point O, if understood, and write E/k meaning that E is an elliptic curve defined over k.

Using the Riemann-Roch theorem it is possible to describe any elliptic curve as the locus in $\mathbb{P}^2$ of a cubic equation with only one point on the line at ∞; see [Sil86] for more details. After a scaling of the coordinates the equation of E is of the following form:

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3$$

and O is the point $(0, 1, 0)$.

We will use the following notation:

k a local field, complete with respect to a discrete valuation v.
$R = \{x \in k \mid v(x) \geq 0\}$, the ring of integers of k.
$R^\times = \{x \in k \mid v(x) = 0\}$, the unit group of R.
$\mathcal{M} = \{x \in k \mid v(x) > 0\}$, the maximal ideal of R.
π a uniformizer for R, i.e. $\mathcal{M} = \pi R$.
K the residue field of R.

Let E be an elliptic curve defined over k. We can assume that all the coefficients in the equation of E lie in a complete discrete valuation ring with perfect residue field and maximal ideal generated by a prime π. Under these hypotheses, E is given by an equation of the following type:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

We define the following quantities:

$$b_2 = a_1^2 + 4 \cdot a_2, \qquad b_4 = a_1 a_3 + 2 \cdot a_4, \qquad b_6 = a_3^2 + 4 \cdot a_6,$$
$$b_8 = a_1^2 a_6 - a_1 a_3 a_4 + 4 \cdot a_2 a_6 + a_2 a_3^2 - a_4^2,$$
$$c_4 = b_2^2 - 24 \cdot b_4, \qquad c_6 = -b_2^3 + 36 \cdot b_2 b_4 - 216 \cdot b_6,$$
$$a_{i,m} = a_i / \pi^m, \qquad \Delta = -b_2^2 b_8 - 8 \cdot b_4^3 - 27 \cdot b_6^2 + 9 \cdot b_2 b_4 b_6, \qquad j = c_4^3 / \Delta.$$

1.5.2. The Group Law. Let (E, O) be an elliptic curve defined over k by a Weierstrass equation. Thus E consists of the point O at infinity and of the points (x, y) satisfying the Weierstrass equation. We can define a composition law on E. Let P, Q be two points on E, let l be the line through P and Q (if $P = Q$, let l be the tangent to E at P) and let R be the third point of intersection of l with E. Let l' be the line through R and O. The third point of intersection between E with l' is denoted by $P + Q$.

The composition law has the following properties:

(1) if a line l intersects E at P, Q, R, then
$$(P + Q) + R = O.$$
(2) $P + O = P$ for all $P \in E$.
(3) $P + Q = Q + P$ for all $P, Q \in E$.
(4) For every point $P \in E$ there exists a point $-P$ such that
$$P + (-P) = O.$$
(5) For every $P, Q, R \in E$ the following holds
$$(P + Q) + R = P + (Q + R).$$

In other words, $(E, +)$ is an abelian group having O as the zero element.

Notice that if E is an elliptic curve defined over k where O is not an inflection point, then it is no longer true that three points on a line add up to O. In this case we have that if P, Q, R are three points of intersection between E and a line then
$$P + Q + R = q,$$
where q is the third point of intersection between E and the tangent to E at O.

1.5.3. Good and Bad Reduction. Let E/k be an elliptic curve. The **reduced curve** $\tilde{E}$ is the image of E via the natural reduction map $R \to R/\pi R$. We can classify E with respect to the type of curve $\tilde{E}$ is. There are the following cases:

(1) if $\tilde{E}$ is non-singular, then E has **good reduction**;
(2) if $\tilde{E}$ has a node, then E has **multiplicative reduction**;
(3) if $\tilde{E}$ has a cusp, then E has **additive reduction**.

In the latter cases we say that E has **bad reduction**. If E has multiplicative reduction, then we say that the reduction is split if the tangents to the node are in $K = R/\pi R$; otherwise we say that the reduction is non-split. We now state a lemma that helps us to understand the reduction type using the valuation of the discriminant Δ.

LEMMA 1.5.4. *Let E/k be an elliptic curve given by a minimal Weierstrass model*
$$E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$
then

(1) *the curve E has good reduction if and only if $v(\Delta) = 0$.*
(2) *the curve E has multiplicative reduction if and only if $v(\Delta) > 0$ and $v(c_4) = 0$. In this case the non-singular $\overline{K}$ points of $\tilde{E}$ form the multiplicative group $\overline{K}^{\times}$.*

(3) *the curve E has additive reduction if and only if $v(\Delta) > 0$ and $v(c_4) > 0$. In this case the non-singular $\overline{K}$ points of $\tilde{E}$ form the additive group $(\overline{K}, +)$.*

PROOF. See [Sil86] $\square$

Now, we can consider an equation defining E as defining a scheme E over $\mathrm{Spec}(R)$. The resulting scheme may not be non-singular, since if E has bad reduction at v, the singular point on the special fiber $\tilde{E}$ of E may be a singular point of the scheme. By resolving the singularity, we obtain a scheme over $\mathrm{Spec}(R)$ whose generic fiber is E/k and whose special fiber is a union of curves over K.

The list of all the possible special fibers is given in Appendix A.

1.6. Tate's Algorithm

Tate's algorithm takes as input an integral model of an elliptic curve over k. The output is the exponent f_v of the conductor, the type of reduction of E with respect to v, given by the Kodaira symbol (see Appendix A), and the index $[E(k) : E^0(k)]$, where $E^0(k)$ denotes the group of k points on E whose reduction is non-singular. Moreover, we can determine whether the integral model is minimal.

1.6.1. The Algorithm. We will describe the algorithm in steps

(1) If π does not divide Δ, we have that $f_v = 0$, the type is I_0 and $c = 1$.
(2) We make a change of coordinates such that π divides a_3, a_4 and a_6.
(3) If π does not divide b_2, then $f_v = 1$, the type is $I_{v(\Delta)}$,
(4) else, if π^2 does not divide a_6, then $f_v = v(\Delta)$, the type is II and $c = 1$,
(5) else, if π^3 does not divide b_8, then $f_v = v(\Delta) - 1$, the type is III and $c = 2$,
(6) else, if π^3 does not divide b_6, then $f_v = v(\Delta) - 2$, the type is IV and $c = 3$,
(7) else, make a change of coordinates such that π divides a_1 and a_2, π^2 divides a_3 and a_4 and π^3 divides a_6. Let q be the polynomial defined as
$$q(t) = t^3 + a_{2,1}t^2 + a_{4,2}t + a_{6,3}.$$
(8) If q has three distinct roots, then $f_v = v(\Delta) - 4$, the type is I_0^* and c is 1+ the number of roots of q in k.
(9) If q has a single and a double root, then $f_v = v(\Delta) - 4 - n$ for some $n > 0$, the type is I_n^* and $c = 2$ or $c = 4$.
(10) The polynomial q has a triple root. We change the coordinates such that the triple root is zero, so that π^2 divides a_1, π^3 divides a_4 and π^4 divides a_6. Let r be the polynomial defined as
$$r(u) = u^2 + a_{3,2}u - a_{6,4}$$
(11) If r has two distinct roots then $f_v = v(\Delta) - 6$, the type is IV^* and $c = 3$ if the roots are in k and $c = 1$ otherwise.
(12) The polynomial r has a double root. We change the coordinates so that it becomes zero. Then π^3 divides a_3 and π^5 divides a_6.
(13) If π^4 does not divide a_4, then $f_v = v(\Delta) - 7$, the type id III^* and $c = 2$,
(14) else, if π^6 does not divide a_6, then $f_v = v(\Delta) - 8$, the type is II^* and $c = 1$,
(15) else the equation is not minimal. We divide all the a_i's by π^i and start again with the new equation.

1.7. Elliptic Surfaces

Let B be a non-singular projective curve defined over an algebraically closed field k of characteristic zero. An **elliptic surface** (defined over k) is an algebraic projective surface $\mathscr{E}$ defined over k, endowed with a fibration $\pi : \mathscr{E} \to B$ such that

- (elliptic fibration) for almost all $t \in B(k)$, $\pi^{-1}(t)$ is a genus-1 curve;
- (section) there exists a k-morphism $\sigma_0 : B \to \mathscr{E}$ such that $\pi \circ \sigma_0 = \mathrm{id}_B$.

We also assume that there exists at least one singular fiber.

An elliptic surface $\pi : \mathscr{E} \to B$ is a **rational elliptic surface** if $\mathscr{E}$ is birational to $\mathbb{P}^2$. In this setting $B = \mathbb{P}^1$.

For every elliptic surface $\pi : \mathscr{E} \to B$, the section σ_0 determines a point O_t on each fiber $E_t = \pi^{-1}(t)$. The couple (E_t, O_t) is an elliptic curve defined over k for almost all $t \in B(k)$.

Moreover, there is another elliptic curve induced by any elliptic surface $\pi : \mathscr{E} \to B$. Let K be the function field $k(B)$. The algebraic surface $\mathscr{E}$ can be seen as an elliptic curve over K. We denote this object by $\mathscr{E}_\mu$ and call it the **generic fiber** of the elliptic surface.
Notice that in case of a rational elliptic surface we have $B = \mathbb{P}^1$, so $k(B) = k(t)$.

The following theorem holds:

Theorem 1.7.1 (Mordell-Weil Theorem for Function Fields). *Let $\pi : \mathscr{E} \to B$ be an elliptic surface defined over an algebraically closed field k of characteristic zero. Let K be the function field $k(B)$. If $\pi : \mathscr{E} \to B$ does not split, then the group $\mathscr{E}_\mu(K)$ is finitely generated.*

Proof. See [Sil94], III. □

In particular, the following equality holds:

$$\mathscr{E}_\mu(K) = \mathbb{Z}^{r_\mu} \oplus T,$$

where T is a torsion group and r_μ is the called the **rank** of the elliptic curve $\mathscr{E}_\mu/K$. We can relate the group of $\mathscr{E}_\mu(K)$ with the group of sections on the corresponding elliptic surface:

Theorem 1.7.2. *Let $\pi : \mathscr{E} \to B$ be an elliptic surface defined over an algebraically closed field k of characteristic zero. Let K be the function field $k(B)$. The set $\mathscr{E}(B/k)$, defined as*

$$\mathscr{E}(B/k) = \{\text{sections } \sigma : B \longrightarrow \mathscr{E} \text{ such that } \sigma \text{ is defined over } k\},$$

is an abelian group. Moreover there is a group isomorphism:

$$\mathscr{E}_\mu(K) \cong \mathscr{E}(B/k).$$

Proof. See [Sil94], III. □

From now on the group $\mathscr{E}(B/k)$ will be called **Mordell-Weil group** of $\mathscr{E}$ and will be denoted with $MW(\mathscr{E})$. Moreover, we will refer to the rank of the generic fiber of $\pi : \mathscr{E} \to B$ as the **(Mordell-Weil) rank** of $\mathscr{E}$.

1.7.3. Construction. We will briefly explain a method to obtain a rational elliptic surface.

Let F and G be two homogeneous cubic polynomials in $k[x_0, x_1, x_2]$, describing two distinct projective plane cubics, with at least one of them being smooth. Consider the rational map

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1 \\ (x, y, z) &\longmapsto (F(x, y, z), G(x, y, z)). \end{aligned}$$

This rational map is not defined exactly at the points where both F and G vanish; by Bézout's theorem this set consists of nine points, counted with multiplicities. These nine points are the base points of a pencil of cubic curves, namely the one generated by F and G. By blowing up these nine points we obtain a rational surface $\mathscr{E}$, together with a morphism $\pi : \mathscr{E} \to \mathbb{P}^1$ whose fibers are genus-1 curves (see [**Mir89**], I.5.1 for more details).

$$\begin{tikzcd} & \mathscr{E} \arrow[dl] \arrow[d, "\pi"] \\ \mathbb{P}^2 \arrow[r, dotted] & \mathbb{P}^1 \end{tikzcd}$$

Every exceptional curve of this blow up is a section of π. We have then constructed a rational elliptic surface.

This construction is general; we have the following theorem by Miranda:

THEOREM 1.7.4. *Let $\pi : \mathscr{E} \to B$ be a rational elliptic surface defined over an algebraically closed field k of characteristic zero. There exists a linear pencil Λ of plane cubics such that the blow-up of $\mathbb{P}^2$ at the base points of Λ is isomorphic to $\mathscr{E}$.*

Notice that the fiber type of the obtained surface depends on the configuration of the base points, thus on the presence of particular members in the pencil of cubics. For example, if the base points of the pencil are three collinear points and three other collinear points counted with multiplicity two, then there is a reducible member in the pencil that splits into a double line and a different line. The induced rational elliptic surface has a fiber of type I_0^* (See [**Mir89**]).

1.8. The Shioda-Tate Formula

This section is devoted to find a relation between the Mordell-Weil group and the Néron-Severi group of an elliptic surface.

Let $\pi : \mathscr{E} \to B$ be an elliptic surface. The points in B such that their pre-image is a non-smooth curve are called **bad places**. The set of all the bad places is denoted with R. The pre-image of a bad place is called a **bad fiber**. If a bad fiber is a reducible curve, it will be called **reducible fiber**.

Let $\pi : \mathscr{E} \to B$ be an elliptic surface with zero-section σ_0. For each $v \in R$, the following equality holds:

$$\pi^{-1}(v) = \Theta_{v,0} + \sum_{i=1}^{m_v-1} \mu_{v,i}\Theta_{v,i},$$

where $\Theta_{v,i}$ $(0 \leq i \leq m_v - 1)$ are the irreducible components of F_v and m_v is the number of components of the fiber. We also define $\Theta_{v,0}$ as the unique component of F_v meeting the zero section; we call $\Theta_{v,0}$ the **zero component** of the fiber $\pi^{-1}(v)$.

THEOREM 1.8.1 (Shioda-Tate Formula). *Let T be the subgroup of $NS(\mathscr{E})$ generated by the zero section σ_0 and all the irreducible components of fibers. We have the following natural isomorphism*

$$MW(\mathscr{E}) \cong \frac{NS(\mathscr{E})}{T}.$$

For a complete description of this isomorphism we refer to [**Shi90**].

1.9. Lattices

We list here some definitions and properties of lattices. For more details, see [**Shi90**]. A **lattice** L is a free $\mathbb{Z}$-module of finite rank, given with a symmetric non-degenerate pairing

$$\langle\ ,\ \rangle : L \times L \longrightarrow \mathbb{Q}.$$

When the pairing takes values in $\mathbb{Z}$, we say that L is an **integral lattice**.

If the rank of L is r and $(x_1, \ldots, x_r)$ is a $\mathbb{Z}$-basis of L we define the **determinant** of L as

$$\det(L) = |\det(\langle x_i, x_j \rangle)|.$$

This number does not depend on the choice of the basis.

We say that a lattice L is **even** if $\langle x, x\rangle \in 2\mathbb{Z}$ for all $x \in L$ and **unimodular** if $\det(L) = 1$.

The **dual lattice** L^* of a lattice L is defined by

$$L^* = \{x \in L \otimes \mathbb{Q} \mid \langle x, y\rangle \in \mathbb{Z} \text{ for all } y \in L\}.$$

Moreover, the following equality holds:

$$\det(L^*) = \frac{1}{\det(L)}.$$

A **sublattice** T of L is a submodule of L such that the restriction of the pairing to $T \times T$ is non-degenerate. The **orthogonal complement** $T^\perp$ of T in L is defined by

$$T^\perp = \{x \in L \mid \langle x, y\rangle = 0 \text{ for all } y \in T\}.$$

For every sublattice T of L of finite index we have the following equality:

$$\det(T) = \det(L) \cdot [L : T].$$

This formula will help us to check if a given set of elements of a lattice is actually a set of generators.

1.9.1. The Néron-Severi Lattice. By [Shi90], thm 3.1 the Néron-Severi group of an elliptic surface becomes an integral lattice with respect to the intersection pairing, called the **Néron-Severi lattice**.
Moreover we have that its rank ρ is given by

$$\rho = r + 2 + \sum_{v \in R} (m_v - 1),$$

where r denotes the rank of the Mordell-Weil group of the elliptic surface.

Let $\pi : \mathscr{E} \to B$ be an elliptic surface. Consider T, the subgroup of $NS(\mathscr{E})$ generated by the zero section and the irreducible components of the fibers of π. By [Shi90], Proposition 2.3, we deduce that T is a sublattice of $NS(\mathscr{E})$, called the **trivial sublattice** of $NS(\mathscr{E})$.

1.9.2. The Mordell-Weil Lattices. By the Shioda-Tate formula (section 1.8), the Mordell-Weil group $MW(\mathscr{E})$ of an elliptic surface $\pi : \mathscr{E} \to B$ is isomorphic to the quotient $NS(\mathscr{E})/T$. We want to define a good pairing on $MW(\mathscr{E})$. The first thing we do is to embed $MW(\mathscr{E})$ into $NS(\mathscr{E}) \otimes \mathbb{Q}$. From [Shi90], Lemma 8.1, for every $P \in MW(\mathscr{E})$, there exists a unique element $\varphi(P)$ of $NS(\mathscr{E}) \otimes \mathbb{Q}$ such that

- $\varphi(P) \equiv (P) \bmod (T \otimes \mathbb{Q})$ and
- $\varphi(P) \perp T$.

Moreover, the map φ is a group homomorphism and $\ker(\varphi) = MW(\mathscr{E})_{\mathrm{tor}}$. For every $P, Q \in MW(\mathscr{E})$ we can define $\langle P, Q\rangle$ as

$$\langle P, Q\rangle = -(\varphi(P) \cdot \varphi(Q)).$$

In this way $\langle\ ,\ \rangle$ is a symmetric bilinear pairing on $MW(\mathscr{E})$, inducing the structure of a positive definite lattice on $MW(\mathscr{E})/MW(\mathscr{E})_{\mathrm{tor}}$. This pairing will be called the **height pairing** and the lattice $(MW(\mathscr{E})/MW(\mathscr{E})_{\mathrm{tor}}, \langle\ ,\ \rangle)$ will be called the **Mordell-Weil lattice** of the elliptic surface.

We have the following explicit formulas to compute the height pairing of any $P, Q \in MW(\mathscr{E})$:

$$\langle P, Q\rangle = \chi + (P \cdot O) + (Q \cdot O) - (P \cdot Q) - \sum_{v \in R} \mathrm{contr}_{v\in R}(P, Q),$$

where χ is the Euler characteristic of the surface, $(P \cdot Q)$ is the intersection number of P and Q and $\mathrm{contr}_v(P, Q)$ gives the local contribution on v. This number can be expressed explicitly, but we first need a rule to label the irreducible components of a reducible fiber.

Let Θ_v be a fiber with m_v simple components. We will denote by $\Theta_{v,0}$ the component intersecting $\sigma_0(B)$. This component is called the **zero component**. All the other components of Θ_v are denoted by $\Theta_{v,i}$ $(i \leq m_v)$, according to the following rule.

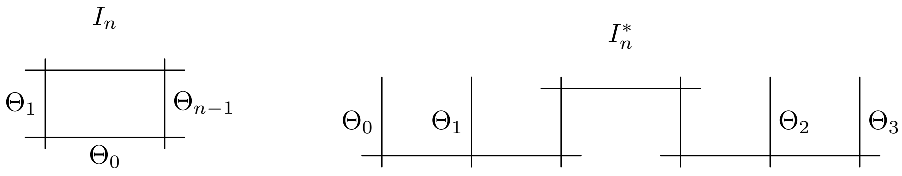


FIGURE 1.2. Enumeration of the components of a fiber, according to the fiber type (from [Shi90]).

Now we can write $\mathrm{contr}_v(P, Q)$ explicitly as

$$\mathrm{contr}_v(P, Q) = \begin{cases} -(A_v^{-1})_{i,j} & \text{if } i, j > 0 \\ 0 & \text{otherwise;} \end{cases},$$

where A_v is the negative definite matrix given by

$$A_v = ((\Theta_{v,i} \cdot \Theta_{v,j})) \qquad 1 \leq i, j \leq m_v - 1.$$

We now give a table listing the possible contribution numbers of P (meeting $\Theta_{v,i}$) and Q (meeting $\Theta_{v,j}$), according to the type of F_v.

Kodaira Symbol	**Dynkin Diagram**	$\mathbf{i = j}$	$\mathbf{i < j}$
I_n, III, IV	A_{n-1}	$\frac{i(n-i)}{n}$	$\frac{i(n-j)}{n}$
I_n^*	D_{n+4}	$\begin{cases} 1 & i = 1 \\ 1 + \frac{n}{4} & i = 2, 3 \end{cases}$	$\begin{cases} \frac{1}{2} & i = 1 \\ \frac{1}{2} + \frac{n}{4} & i = 2 \end{cases}$
III^*	E_7	$\frac{3}{2}$	$-$
IV^*	E_6	$\frac{4}{3}$	$\frac{2}{3}$

TABLE 1.1. The contribution terms for any fiber type (from [Shi90]).

We now define a subgroup of $MW(\mathscr{E})$ denoted by $MW(\mathscr{E})^0$:

$$MW(\mathscr{E})^0 = \{P \in MW(\mathscr{E}) \mid P \text{ meets } \Theta_{v,0} \text{ for all } v \in R\}.$$

This subgroup is torsion-free (see [Shi90]) and can be viewed as a lattice with respect to the height pairing. This lattice is a positive definite even lattice, called the **narrow Mordell-Weil lattice**.
Moreover we have the following equality:

$$\det(MW(\mathscr{E})^0) = \frac{\det(NS(\mathscr{E})) \cdot [MW(\mathscr{E}) : MW(\mathscr{E})^0]^2}{\det(T)}.$$

1.9.3. Results on Rational Elliptic Surfaces. Let $\pi : \mathscr{E} \to B$ be a rational elliptic surface. In this case the Néron-Severi lattice $NS(\mathscr{E})$ is unimodular of rank 10 and the Mordell-Weil lattice $MW(\mathscr{E})/MW(\mathscr{E})_{\mathrm{tor}}$ is the dual of the narrow Mordell-Weil lattice $MW(\mathscr{E})^0$.

The relation between the narrow Mordell-Weil lattice, the trivial lattice and the Mordell-Weil group becomes the following:

$$\det(MW(\mathscr{E})^0) = \frac{1 \cdot [MW(\mathscr{E}) : MW(\mathscr{E})^0]^2}{\det(T)}.$$

These conditions give a criterion to decide whether a set of elements in the Mordell-Weil group is a basis for the Mordell-Weil lattice as soon as we know the structure of the trivial lattice and the narrow Mordell-Weil lattice.

CHAPTER 2

Construction of Rational Elliptic Surfaces with Rank 4

2.1. Reducible Fibers on Rational Elliptic Surfaces with Rank 4

Let $\pi : \mathscr{E} \to \mathbb{P}^1$ be a rational elliptic surface with Mordell-Weil rank 4. Since the rank of the Néron-Severi group $NS(\mathscr{E})$ is 10 (see [Shi90]) and the rank of the Mordell-Weil group is 4, we can use the Shioda-Tate formula (section 1.8) together with the results on the Néron-Severi lattice

$$\mathrm{rank}(NS(\mathscr{E})) = \mathrm{rank}(MW(\mathscr{E})) + 2 + \sum_{v \in R} (m_v - 1)$$

to deduce the contribution given by the bad fibers. We have that

$$\sum_{v \in R} (m_v - 1) = 4.$$

So, only the following cases can occur:

(1) $m_v = 5$: there is a unique bad fiber, with 5 components;
(2) $m_{v_1} = 4, \quad m_{v_2} = 2$: there are two bad fibers, one with 4 components and the other with 2 components;
(3) $m_{v_1} = 3, \quad m_{v_2} = 3$: there are two bad fibers, both with 3 components;
(4) $m_{v_1} = 3, \quad m_{v_2} = m_{v_3} = 2$: there are three bad fibers, one with 3 components and the others with 2 components;
(5) $m_{v_1} = \cdots = m_{v_4} = 2$: there are four bad fibers, all with 2 components.

We will now state a crucial theorem, that helps us to understand each case listed above:

Theorem 2.1.1 (Oguiso-Shioda). *The following table summarizes the possible lattice structures for the Mordell-Weil group of a rational elliptic surface of rank 4. We denote by T' the lattice associated with the reducible fibers.*

T'	$\det(T')$	$MW(\mathscr{E})^0$	$MW(\mathscr{E})$
A_4	5	A_4	A_4^*
D_4	4	D_4	D_4^*
$A_3 \oplus A_1$	8	$A_3 \oplus A_1$	$A_3^* \oplus A_1^*$
$A_2^{\oplus 2}$	9	$A_2^{\oplus 2}$	$A_2^{*\oplus 2}$
$A_2 \oplus A_1^{\oplus 2}$	12	$\begin{pmatrix} 4 & 1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 2 \end{pmatrix}$	$\frac{1}{6}\begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 5 & 3 & 1 \\ 0 & 3 & 6 & 3 \\ -1 & 1 & 3 & 5 \end{pmatrix}$
$A_1^{\oplus 4}$	16	$A_1^{\oplus 4}$	$A_1^{*\oplus 4}$
$A_1^{\oplus 4}$	16	D_4	$D_4^* \oplus \mathbb{Z}/2\mathbb{Z}$

PROOF. See [OS91]. □

A rational elliptic surface $\mathscr{E}$ is isomorphic to the blow-up of a linear pencil Λ of cubics (theorem 1.7.4). In particular for any bad fiber F_v of $\mathscr{E}$, Λ contains the image of F_v in $\mathbb{P}^2$, which is a curve of degree 3 (since Λ is a pencil of cubics). The configuration of the base points of Λ must be compatible with the presence of this member.

For a complete list of all the possible images in $\mathbb{P}^2$ of every fiber that can appear in a rational elliptic surface of rank 4 see Appendix B.

2.2. Technique

For each possible lattice structure of a rational elliptic surface with rank four, we first find nine points on the plane that are the base points of a linear pencil of cubics that induce a rational elliptic surface with the given lattice structure.

If the pencil does not obviously have a smooth member, we show that it actually has a smooth member.

We then find all the non-smooth members of the pencil, in order to be sure that we are constructing the correct rational elliptic surface.

Later we find the configuration of the exceptional curves above the base points with respect to each reducible fiber; in other words, we look at the component of the fiber the exceptional curve is meeting. This is done in order to compute the height matrix of the exceptional curves above the base points.

Finally, we check that the determinant of the obtained matrix is equal to the determinant of the Mordell-Weil group of the induced rational elliptic surface. This implies that the chosen exceptional curves generate the Mordell-Weil group of the surface.

2.3. A Unique Reducible Bad Fiber

Since the unique bad fiber has 5 components, it must be either a fiber of type I_5 or of type I_0^* (See Appendix A). We will analyze these two cases separately.

2.3.1. A Fiber of Type I_5. From section B.2, we know that in order to have a fiber of type I_5 the pencil of cubics must contain a member of one of the following forms:

(1) A nodal cubic such that the singular point is a base point with multiplicity 5.
(2) A reducible cubic, split into a line and an irreducible conic, such that either
    (a) the intersection points are base points with multiplicity 3 and 2 respectively, or
    (b) only one intersection point is a base point and has multiplicity 4.
(3) A reducible cubic, split into 3 non-concurrent lines, such that
    (a) one of the three intersection points between the lines is not a base point, while the others are base points with multiplicity 2.
    (b) one intersection point is a base point with multiplicity 3 and the remaining intersection points are not base points.

2.3.1.1. *Construction.* We will construct an elliptic surface of rank 4 from a linear pencil of cubics as in $(3)(a)$.

Let E be a non-singular plane cubic. Take a line l_1 on the plane such that it intersects the curve E at three distinct points p_0, p_1, p_2. Take a line l_2, passing through p_0 and such that it intersects E in two other different points p_3 and p_4. Suppose that the lines passing through p_4 and p_1 and the one through p_4 and p_2 are not tangent to E at p_4. Now, let l_3 be a line passing through p_4, and two other different points p_5 and p_6 such that p_5, p_6 are each non-collinear with any other two base points and such that the lines through p_0 and one of them are not tangent to E at p_0.

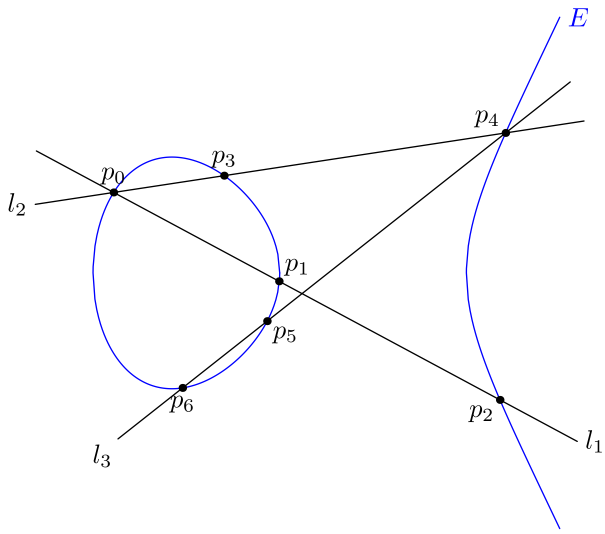


FIGURE 2.1. Configuration of the base points of a pencil of cubics inducing a rational elliptic surface with a fiber of type I_5.

Consider the linear pencil of cubics Λ, generated by E and $R = l_1 l_2 l_3$. This pencil can be described as the pencil of cubics passing through $p_0, \ldots, p_6$ with prescribed tangent at p_0 and p_4 (the tangent of E at p_0 and the tangent of E at p_4, respectively). We now describe all the singular members in Λ.

The base points with multiplicity 2 cannot be singular points of any irreducible member in Λ, by Bézout's theorem. Moreover, the unique reducible member is R. We will show this last statement in detail. If a cubic C in Λ is reducible, it contains a conic Q, possibly reducible, and a line l.

Suppose by contradiction that the conic Q is irreducible and it is not tangent to E at p_0 nor at p_4; then the line l needs to pass through both p_0 and p_4, so $l = l_2$. Since l_2 is not tangent to E, then Q should pass through all base points, except p_3. This contradicts the fact that Q was irreducible. So Q is reducible or it is tangent to E at p_0 or p_4.

Suppose by contradiction that it is irreducible and tangent to E at p_0 or p_4. If Q is tangent to only p_i, $i \in \{0, 4\}$, then the line l should pass through p_{4-i}. This implies that p_3 is on Q, otherwise l is l_2 and Q would split. Given that p_3 is on Q, then l must be tangent to E at p_{4-i}. By the hypotheses on the tangents to E at p_{4-i} we have that l does not pass through any other base point. So Q should pass through p_i, p_1, p_2, p_3, p_5, p_6. This is impossible, since three of them are on a line and Q was supposed to be irreducible. Then Q is tangent to E at both p_0 and p_4. Since Q is irreducible, then it cannot pass through p_3. For the same reason it cannot pass through both p_1 and p_2 and through both p_5 and p_6. So three points among p_3, p_1, p_2, p_5, p_6 must be on l. This is again impossible, since we supposed that they are not collinear. This implies that Q is reducible.

Thus, C splits into the product of three lines and by the hypotheses on the collinearity on the base points, C must be R.

Notice that R corresponds to an I_5 fiber in the rational elliptic surface given by the blow-up of $\mathbb{P}^2$ at the base points of Λ.

We denote by $P_0, \dots, P_6$ the (-1)-curves above $p_0, \dots, p_6$.
Let P, Q be two elements in the Mordell-Weil group of the rational elliptic surface given by the blow-up of $\mathbb{P}^2$ at the base points of Λ. Using the Contribution Table in section 1.9.2, we find that, if P meets the component Θ_i, the contribution of the I_5 fiber to $\langle P, P\rangle$ is given by:

$$\mathrm{contr}(P) = \frac{i(5-i)}{5}.$$

If P meets the component Θ_i and Q meets the component Θ_j with $i \leq j$, the contribution of the I_5 fiber to $\langle P, Q\rangle$ is given by:

$$\mathrm{contr}(P, Q) = \frac{i(5-j)}{5}.$$

We set P_0 as the zero section. We now check the intersections between the components of the fiber of type I_5 and the curves $P_1, \dots, P_6$, using the technique described in section B.1. Since p_0 is on l_1, the line where p_1 and p_2 lie, the curves P_1 and P_2 must intersect either Θ_1 or Θ_4, say Θ_1. Since p_0 is also on l_2, the point P_3 must intersect either Θ_1 or Θ_4, but not the same as the one intersecting P_1 and P_2. So, P_3 must intersect Θ_4. Since p_4 is blown up twice, the associated (-1)-curve P_4 does not intersect the same component that P_3 intersects, and must intersect Θ_3. With a similar argument P_5 and P_6 intersect Θ_2. The configuration of the exceptional curves in the I_5 fiber is the following:

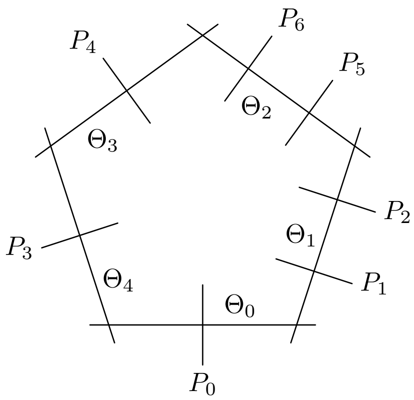


FIGURE 2.2. Configuration of the exceptional curves above the base points of a pencil of cubics as in figure 2.1, inducing a rational elliptic surface with a fiber of type I_5.

Using the above formulas we have:

$$(\mathrm{contr}(P_i, P_j))_{i,j} = \begin{pmatrix} 4/5 & 4/5 & 1/5 & 2/5 & 3/5 & 3/5 \\ 4/5 & 4/5 & 1/5 & 2/5 & 3/5 & 3/5 \\ 1/5 & 1/5 & 4/5 & 3/5 & 2/5 & 2/5 \\ 2/5 & 2/5 & 3/5 & 6/5 & 4/5 & 4/5 \\ 3/5 & 3/5 & 2/5 & 4/5 & 6/5 & 6/5 \\ 3/5 & 3/5 & 2/5 & 4/5 & 6/5 & 6/5 \end{pmatrix}.$$

We consider the matrix given by the heights of P_1, P_3, P_4 and P_5. The height matrix is the following:

$$A_{I_5} = \begin{pmatrix} 6/5 & 4/5 & 3/5 & 2/5 \\ 4/5 & 6/5 & 2/5 & 3/5 \\ 3/5 & 2/5 & 4/5 & 1/5 \\ 2/5 & 3/5 & 1/5 & 4/5 \end{pmatrix}.$$

According to theorem 2.1.1, the Mordell-Weil lattice of the induced surface is isomorphic to A_4^*, in particular it has determinant equal to $1/5$. Since the determinant of the matrix A_{I_5} is equal to $1/5$, the elements P_1, P_3, P_4 and P_5 generate the full Mordell-Weil group of the rational elliptic surface.

2.3.1.2. *Construction.* We will construct an elliptic surface of rank 4 from a linear pencil of cubic as in $(2)(a)$.

Let E be a non-singular plane cubic curve. Take a point p_0 in E such that the tangent t to E at p_0 meets E in a point $q \neq p_0$ and let l be a line passing through p_0, non-tangent to E. Then there exist two distinct points p_4 and p_5 such that E and l meet at p_0, p_4 and p_5. Now, take a conic Q passing through p_4 and p_0, not passing through q, tangent to E at p_0 and not tangent to E in any other point of intersection. Then, there exist p_1, p_2 and p_3 such that E and Q meet at p_0, p_1, p_2, p_3, p_4 and share the tangent at p_0. We choose the conic such that p_5 is not collinear with any other couple of points p_i, p_j except p_0, p_4 and the tangent to E at p_4 does not meet any other p_i.

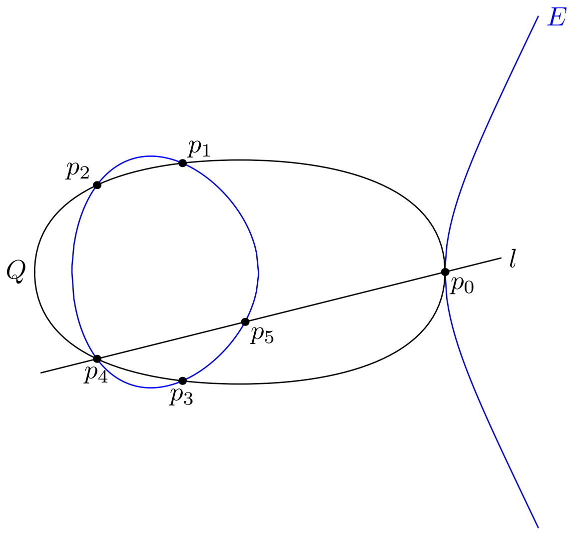


FIGURE 2.3. Configuration of the base points of a pencil of cubics inducing a rational elliptic surface with a fiber of type I_5.

Consider the pencil of cubics Λ generated by E and $R = Ql$. We now describe all the singular members in Λ.

The base points p_0 and p_4 cannot be singular points of any irreducible member in Λ, by Bézout theorem. We now want to show that R is the only reducible member. Since the points p_0, p_4, p_5 are the only base points in a line, all the reducible members in Λ split into a line and an irreducible conic. Since q was not a base point, the line cannot be tangent to E at p_0. This means that the conic is tangent to E at p_0, thus the intersection multiplicity at p_0 between E and the conic is at least 2. In fact, it is exactly 2. If, by contradiction, the intersection multiplicity between E and the conic at p_0 was greater or equal to 3, then the conic would meet E at three other points at most; then the remaining points of intersection with E would not be collinear, contradicting the assumption they were on a line. This means that the intersection multiplicity between E and the conic is exactly 2 and the line passes through p_0. The only line and conic fitting these hypotheses are Q and l.

As in the previous construction, we denote by P_i the (-1)-curve above p_i and set P_0 as the zero section. The configuration of the exceptional curves on the I_5 fiber is the following (notice that all the cubics are tangent to Q at p_0):

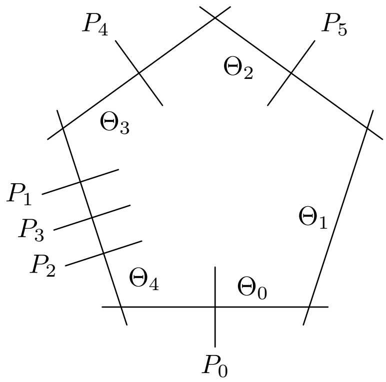


FIGURE 2.4. Configuration of the exceptional curves above the base points of a pencil of cubics inducing a rational elliptic surface with a fiber of type I_5.

2.3.1.3. *Equivalence between the Constructions.* We will show that there are birational maps that change the linear pencil of cubics described in (3)(a) into the one described in (2)(a) and vice versa.

First we set $\mathscr{E}$ as the rational elliptic surface described in construction (2)(a). We will use the same notation as in that section. Since p_0 is a base point of Λ of multiplicity 3, there are three curves above it in $\mathscr{E}$: the (-1)-curve P_0, a (-2)-curve P_0' and another curve. After contracting P_0, we get a new surface where the image of the curve P_0' is a (-1)-curve and can be contracted itself. We will denote by $\mathscr{E}'$ the surface obtained contracting first P_0 and then the image of P_0'. Since p_4 is a base point of Λ of multiplicity 2, there are two curves above it in $\mathscr{E}$: the (-1)-curve P_4 and a (-2)-curve P_4'. Their images in $\mathscr{E}'$ are isomorphic to the original curves, so we will denote them with the same letters. After contracting $P_4 \in \mathscr{E}'$, we have that the image of P_4' can be contracted. We will denote by $\mathscr{E}''$ the surface obtained from $\mathscr{E}'$ contracting P_4 and subsequently the image of P_4'. The image in $\mathscr{E}''$ of the each other exceptional curve P_i is still isomorphic to P_i, so we will denote it with the same letter. Now, let E_1 be the strict transform on $\mathscr{E}''$ of the line passing through p_0 and p_1, let E_2 be the strict transform on $\mathscr{E}''$ of the line passing through p_0 and p_2 and let E_3 be the strict transform on $\mathscr{E}''$ of the line passing through p_1 and p_2. The (-1)-curves P_3, P_5, E_1, E_2 and E_3 do not intersect each other in $\mathscr{E}''$ and contracting them we find a linear pencil of cubics as in (3)(a).

On the other hand, let $\mathscr{E}$ be the rational elliptic surface described in (3)(a). We will use the same notation as in that section. Since p_0 is a base point of Λ of multiplicity 2, there are two curves above it in $\mathscr{E}$: the (-1)-curve P_0 and a (-2)-curve P_0'. After contracting P_0, we get a new surface where the image of the curve P_0' is a (-1)-curve and can be contracted itself. We will denote by $\mathscr{E}'$ the surface obtained contracting first P_0 and then the image of P_0'. Since p_4 is a base point of Λ of multiplicity 2, there are two curves above it in $\mathscr{E}$: the (-1)-curve P_4 and a (-2)-curve P_4'. Their images in $\mathscr{E}'$ are isomorphic to the original curves, so we will denote them with the same letters. After contracting $P_4 \in \mathscr{E}'$, we have that the image of P_4' can be contracted. We will denote by $\mathscr{E}''$ the surface obtained from $\mathscr{E}'$ contracting P_4 and subsequently the image of P_4'. The image in $\mathscr{E}''$ of the each other exceptional curve P_i is still isomorphic to P_i, so we will denote it with the same letter. Now, let E_1 be the strict transform on $\mathscr{E}''$ of the line passing through p_5 and p_1, let E_2 be the strict transform on $\mathscr{E}''$ of the line passing through p_5 and p_2 and let L_1 be the strict transform on $\mathscr{E}''$ of l_1. The (-1)-curves P_3, P_4, P_6, E_1, E_2 and L_1 do not intersect each other in $\mathscr{E}''$ and contracting them we find a linear pencil of cubics as in $(3)(a)$.

Theorem 2.3.2. *Let $\mathscr{E}$ be a rational elliptic surface with Mordell-Weil rank four and $MW(\mathscr{E}) \cong A_4^*$. Then $\mathscr{E}$ arises from a linear pencil of cubic curves as in construction* $(3)(a)$.

Proof. We must show that all the possible constructions of a rational elliptic surface of rank four with a fiber of type I_5 are equivalent to $(3)(a)$. We know from section B.2 that there are five constructions and we just showed that $(3)(a)$ and $(2)(a)$ are equivalent. With similar arguments one can show that all the constructions are equivalent.

We now show that $(3)(a)$ and $(3)(b)$ are equivalent.
Suppose we are working in the settings of $(3)(a)$. Let $\mathscr{E}$ be the induced rational elliptic surface. We will use the same notation as in the construction we already made. Let $l_{i,j}$ be the line passing through p_i and p_j. We can obtain a pencil as in $(3)(b)$ by contracting both the exceptional curves above p_0, one of the two exceptional curves above p_4, the strict transforms of the lines $l_{1,3}$, of $l_{1,4}$ and of $l_{3,4}$ and the curves P_2, P_5 and P_6.

Conversely, suppose we are working in the settings of $(3)(b)$. Let $\mathscr{E}$ be the induced rational elliptic surface. Let $p_0, \dots, p_6$ be the base points of the pencil. Let p_0 be the point of multiplicity three, l_3 the line of the reducible member not passing through p_0, l_1 the line tangent to the smooth members at p_0 and l_2 the last line composing the reducible member. Let p_1 be a base point in l_2 and p_2 a base point in l_3. Let $l_{i,j}$ be the line passing through p_i and p_j. We can obtain a pencil as in $(3)(a)$ by contracting two of the three exceptional curves above p_0, the strict transforms of $l_{0,1}$, of $l_{0,2}$ and of $l_{1,2}$ and every exceptional curve above p_i, $i \neq 0, 1, 2$.

We now show that $(2)(b)$ and $(3)(b)$ are equivalent.
Suppose we are working in the settings of $(2)(b)$. Let $\mathscr{E}$ be the induced rational elliptic surface. Let $p_0, \dots, p_5$ be the base points of the pencil. Let p_0 be the point of multiplicity four, p_1 and p_2 two base points on the conic belonging to the cubic inducing the fiber of type I_5 and $l_{i,j}$ the line passing through p_i and p_j. We can obtain a pencil as in $(3)(b)$ by contracting three of the four exceptional curves above p_0, the strict transforms of the lines $l_{0,1}$, $l_{0,2}$ and $l_{1,2}$ and every exceptional curve above p_i, $i \neq 0, 1, 2$.

Conversely, suppose we are working in the settings of $(3)(b)$. Let $\mathscr{E}$ be the induced rational elliptic surface. Let $p_0, \dots, p_6$ be the base points of the pencil. Let p_0 be the point of multiplicity three, l_3 the line of the reducible member not passing through p_0, l_1 the line tangent to the smooth members at p_0 and l_2 the last line composing the reducible member. Let p_1 and p_2 be base points in l_2 and p_3 a base point in l_3. Let $l_{i,j}$ be the line passing through p_i and p_j. We can obtain a pencil as in $(2)(b)$ by contracting all the exceptional curves above p_0 (three in total), the strict transforms of the lines $l_{1,3}$, $l_{2,3}$ and l_2 and every exceptional curve above p_i, $i \neq 0, 1, 2, 3$.

We now show that (1) and $(2)(b)$ are equivalent.
Suppose we are working in the settings of (1). Let $\mathscr{E}$ be the induced rational elliptic surface. Let $p_0, \dots, p_4$ be the base points of the pencil. Let p_0 be the point of multiplicity five, p_1 and p_2 two base points and $l_{i,j}$ the line passing through p_i and p_j. We can obtain a pencil as in $(2)(b)$ by contracting four of the five

exceptional curves above p_0, the strict transforms of the lines $l_{0,1}$, $l_{0,2}$ and $l_{1,2}$ and every exceptional curve above p_i, $i \neq 0, 1, 2$.

Conversely, suppose we are working in the settings of (2)(b). Let $\mathscr{E}$ be the induced rational elliptic surface. Let $p_0, \ldots, p_5$ be the base points of the pencil. Let p_0 be the point of multiplicity four, p_1 and p_2 two base points on the line, p_3 a base point on the conic and $l_{i,j}$ the line passing through p_i and p_j. We can obtain a pencil as in (1) by contracting all the exceptional curves above p_0 (four in total), the strict transforms of the lines $l_{1,2}$, $l_{1,3}$ and $l_{2,3}$ and every exceptional curve above p_i, $i \neq 0, 1, 2, 3$.

This concludes the proof. □

2.3.3. A Fiber of Type I_0^*. From section B.2, in order to obtain a fiber of type I_0^* the pencil of cubics must contain a member of the following forms:

(1) A cuspidal cubic, such that the cusp is a base point with multiplicity 5
(2) A reducible cubic, given by the product of an irreducible conic and a line tangent to that conic, such that the intersection point between them is a base point with multiplicity 4
(3) A reducible cubic, given by the product of three concurrent lines, such that the intersection point between those lines is a base point with multiplicity 3
(4) A reducible cubic, given by the product of a double line and a different line, such that the intersection point is not a base point.

2.3.3.1. *Construction.* We will construct an elliptic surface with rank 4 as in (3).

Let E be a smooth cubic curve. Consider a point p_0 on it and let l_1 be a line passing through p_0 intersecting E at two other distinct points p_1 and p_2. Take now a different line l_2 passing through p_0 and two other distinct points p_3 and p_4. Let l_3 be a line passing through p_0 and two other distinct points p_5 and p_6, not collinear with any other two different base points (except p_0, p_5, p_6), and such that the points $p_1, \ldots, p_6$ do not lie on a conic.

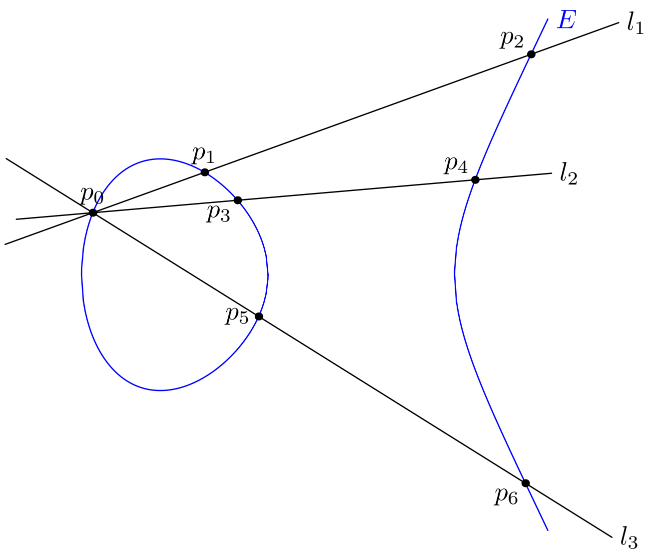


FIGURE 2.5. Configuration of the base points of a pencil of cubics inducing a rational elliptic surface with a fiber of type I_0^*.

Consider the pencil of cubics Λ generated by E and $R = l_1 l_2 l_3$. The point p_0 cannot be the singular point of any irreducible member of Λ, by Bézout's theorem. We will now show that the cubic R is the only reducible member. Suppose that there is another reducible member in the pencil. It cannot split into a line and an irreducible conic: by the assumption that $p_1, \ldots, p_6$ do not lie in a conic, the conic should pass through p_0 and at least two other collinear points, but this is not possible for an irreducible conic. Then the cubic must split into 3 lines. Since the points $\{p_{2k-1}, p_{2k}\}_{k=1,2,3}$ are collinear with p_0 and no other combination of three points is on a line, the reducible member is R.

Notice that the unique reducible member R in Λ corresponds to an I_0^* fiber in $\mathscr{E}$.

We denote with $P_0, \ldots, P_6$ the (-1)-curves above $p_0, \ldots, p_6$.
Let P, Q be two elements in the Mordell-Weil group of the rational elliptic surface given by the blow-up of $\mathbb{P}^2$ at the base points of Λ. Using the Contribution Table in section 1.9.2, we find that, if P meets the component Θ_i and Q meets the component Θ_j with $i \leq j$, the contribution of the I_0^* fiber to $\langle P, Q \rangle$ is given by:

$$\mathrm{contr}(P, Q) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i < j. \end{cases}$$

We set P_0 as the zero section. Using the technique described in section B.1, we can deduce that P_i and P_j, $(i < j)$, intersect the same component if and only if $i = 1, 3, 5$ and $j = i + 1$ because the blowing-up of p_0 separates points on different lines. So, we get the following matrix:

$$(\mathrm{contr}(P_i, P_j))_{i,j} = \begin{pmatrix} 1 & 1 & 1/2 & 1/2 & 1/2 & 1/2 \\ 1 & 1 & 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1 & 1 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1 & 1 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1 & 1 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1 & 1 \end{pmatrix}$$

This allows us to compute the height pairing of the points $P_1, \ldots, P_6$, since the sections above the base points do not intersect each other. We will consider the matrix given by the height pairing of P_1, P_2, P_3 and P_5. The height matrix is then the following:

$$A_{I_0^*} = \begin{pmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \end{pmatrix}.$$

According to theorem 2.1.1, the Mordell-Weil lattice of the induced surface is isomorphic to D_4^*, in particular it has determinant equal to $1/4$. Since the determinant of the matrix $A_{I_0^*}$ is equal to $1/4$, the elements P_1, P_2, P_3 and P_5 generate the full Mordell-Weil group of the rational elliptic surface.

2.3.3.2. *Construction.* We will construct an elliptic surface with rank 4 as in (4).
Let E be a smooth cubic plane curve. Let l_1 be a line intersecting E at three distinct points p_0, p_1, p_2. Let t_i be the tangent line to E at p_i $i = 0, 1, 2$. Let l_2 be another line, not passing through any p_i $(i = 0, 1, 2)$, meeting E at three distinct points p_3, p_4, p_5, such that at least one of them does not lie in any t_i.

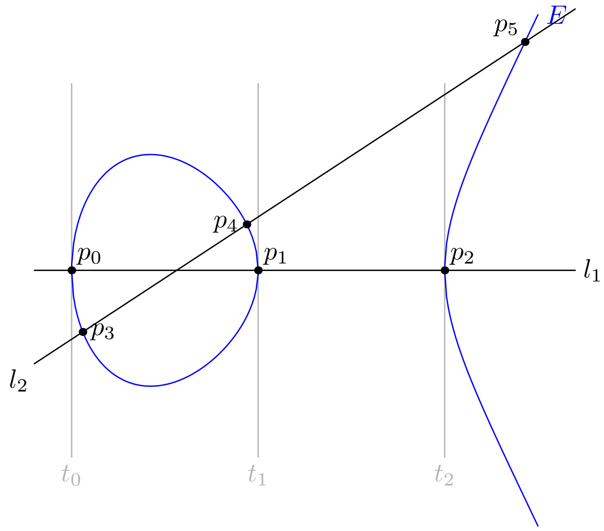


FIGURE 2.6. Configuration of the base points of a pencil of cubics inducing a rational elliptic surface with a fiber of type I_0^*.

Consider the pencil of cubics Λ generated by E and $l_1^2 l_2$. The points p_0 and p_4 cannot be the singular point of any irreducible member of Λ, by Bézout's theorem. The only reducible member is $l_1^2 l_2$: if a cubic in Λ splits into a conic and a line, the line must be either l_1 or l_2, for the collinearity properties we required; hence the conic must also split and the reducible cubic is $l_1^2 l_2$.

2.3.3.3. *Equivalence between the Constructions.* We want to show that the constructions above lead to the same rational elliptic surface; that is, there exist birational maps that change the rational elliptic surface given in one construction into the rational elliptic surface given by the other construction.

First, let $\mathscr{E}$ be the rational elliptic surface described in construction (4). We will use the same notation as in that section. Since p_0 is a base point of Λ of multiplicity 2, there are two curves above it: the (-1)-curve P_0 and a (-2)-curve P_0'. After contracting P_0, we get a new surface where the image of the curve P_0' is a (-1)-curve and can be contracted itself. We will denote by $\mathscr{E}'$ the surface obtained contracting first P_0 and then the image of P_0'. Each image in $\mathscr{E}'$ of the exceptional curve P_i is isomorphic to P_i itself, so it will be called P_i again. Let E_1 be the strict transform on $\mathscr{E}'$ of the line passing through p_1 and p_5, E_2 be the strict transform on $\mathscr{E}'$ of the line passing through p_2 and p_5 and L_1 be the strict transform on $\mathscr{E}'$ of l_1. The (-1)-curves $L_1, P_1, P_2, P_3, P_4, E_1, E_2$ do not intersect each other and contracting them we find a linear pencil of cubics as in (3).

On the other hand, let $\mathscr{E}$ be the rational elliptic surface described in construction (3). We will use the same notation as in that section. Since p_0 is a base point of Λ of multiplicity 3, there are three curves above it: the (-1)-curve P_0, a (-2)-curve P_0' and another curve we are not interested in. After contracting P_0, we get a new surface where the image of the curve P_0' is a (-1)-curve and can be contracted itself. We will denote by $\mathscr{E}'$ the surface obtained contracting first P_0 and then the image of P_0'. Each image in $\mathscr{E}'$ of the exceptional curve P_i is isomorphic to P_i itself, so it will be called P_i again. Let E_1 be the strict transform on $\mathscr{E}'$ of the line passing through p_2 and p_3. Let L_1 be the strict transform of l_1 and let L_2 be the strict transform of l_2. The (-1)-curves $P_1, P_3, P_5, P_6, L_1, L_2$ and E_1 do not intersect each other and contracting them we get a linear pencil of cubics as in (4).

THEOREM 2.3.4. *Let $\mathscr{E}$ be a rational elliptic surface with Mordell-Weil rank four and $MW(\mathscr{E}) \cong D_4^*$. Then $\mathscr{E}$ arises from a linear pencil of cubic curves on as in construction* (3).

PROOF. We must show that all the possible constructions of a rational elliptic surface of rank four with a fiber of type I_0^* are equivalent to (3). We know from section B.2 that there are four constructions; we just showed that (3) and (4) are equivalent. With similar arguments one can show that all the possible constructions are equivalent.

We now show that (2) and (3) are equivalent.

Suppose we are working in the settings of (2). Let $\mathscr{E}$ be the induced rational elliptic surface. Let $p_0, \ldots, p_5$ be the base points of the pencil. Let p_0 be the base point of multiplicity four, p_1 and p_2 two base points on the conic belonging to the reducible member inducing the fiber of type I_0^* and $l_{i,j}$ the line passing through p_i and p_j. We can obtain a pencil as in (3) by contracting three of the four exceptional curves above p_0, the strict transforms of the lines $l_{0,1}$, $l_{0,2}$ and $l_{1,2}$ and every exceptional curve above p_i, $i \neq 0, 1, 2$.

Conversely, suppose we are working in the settings of (3). Let $\mathscr{E}$ be the induced rational elliptic surface. Let $p_0, \ldots, p_6$ be the base points of the pencil. Let p_0 be the base point of multiplicity three, p_1 and p_2 two base points on the same line l, p_3 a base point not in l and $l_{i,j}$ the line passing through p_i and p_j. We can obtain a pencil as in (2) by contracting all the tree exceptional curves above p_0, the strict transforms of the lines $l_{1,3}$, $l_{2,3}$ and l and every exceptional curve above p_i, $i \neq 0, 1, 2, 3$.

We now show that (1) and (2) are equivalent.

Suppose we are working in the settings of (1). Let $\mathscr{E}$ be the induced rational elliptic surface. Let $p_0, \ldots, p_4$ be the base points of the pencil. Let p_0 be the base point of multiplicity five, p_1 and p_2 two other base points and $l_{i,j}$ the line passing through p_i and p_j. We can obtain a pencil as in (2) by contracting four of the five exceptional curves above p_0, the strict transforms of the lines $l_{0,1}$, $l_{0,2}$ and $l_{1,2}$ and every exceptional curve above p_i, $i \neq 0, 1, 2$.

Conversely, suppose we are working in the settings of (2). Let $\mathscr{E}$ be the induced rational elliptic surface. Let $p_0, \ldots, p_5$ be the base points of the pencil. Let p_0 be the base point of multiplicity four, p_1 the base point on the line l, tangent to the conic at p_0 and let $l_{i,j}$ the line passing through p_i and p_j. We can obtain the pencil as in (1) by contracting four exceptional curves above p_0, the strict transform of l, and the strict transforms of all $l_{1,i}$ with $i > 1$.

This concludes the proof.

□

2.4. Two Reducible Fibers

According to section 2.1, there are two cases:

(1) one bad fiber has 4 components and the other has 2 components;
(2) both bad fibers have 3 components.

We will start with the first case: from Appendix A we know that the fiber with 4 components can only be of type I_4, while the fiber with two components can be either of type I_2 or of type III.

2.4.1. A Fiber of type I_4 and a Fiber with two Components. From section B.3, in order to have a fiber of type I_4, the pencil of cubics must have a member of one of the following forms:

(1) A nodal cubic such that the singular point is a base point with multiplicity 4.
(2) A reducible cubic, split into a line and an irreducible conic, such that either
 (a) the intersection points are base points with multiplicity 2 and 2 respectively, or
 (b) only one intersection point is a base point and has multiplicity 3.
(3) A reducible cubic, split into 3 non-concurrent lines, such that one intersection point between the lines is base point with multiplicity 2 and the other intersection points are not base points.

From section B.5, in order to have a fiber with two components, the pencil of cubics must have a member of one of the following forms: and a member of one of the following forms:

(1′) A rational irreducible cubic such that the singular point is a base point with multiplicity 2,
(2′) A reducible cubic, split into an irreducible conic and a line, such that the intersection points between them are not base points.

2.4.1.1. *Construction.* We will construct an elliptic surface of rank 4 from a linear pencil of cubics as in $(3) + (2')$.

Let Q be an irreducible conic and let l be a line. Let q_1 and q_2 be the intersection points between Q and l ($q_1 = q_2$ can happen). Let l_1 be a line such that the intersection points with Q are p_0 and $p_1 \neq p_0$ and the intersection point with l is $p_2 \neq q_i$, $i = 1, 2$. Let l_2 be a line passing through p_0 such that the intersection points with Q are p_0 and $p_3 \neq p_0$ and the intersection point with l is $p_4 \neq q_i$, $i = 1, 2$. Let l_3 be a line such that the intersection points with Q are p_5, p_6 and the intersection point with l is $p_7 \neq q_i$, $i = 1, 2$. We will choose l_3 such that p_5, p_6, p_7 are not equal to any other point p_i and they are not collinear to any couple of other points p_i. Let t be the tangent to Q at p_0.

Consider the pencil of cubics Λ passing through $p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7$, with prescribed tangent t at p_0. This pencil contains a smooth member, as shown in section C.2. The point p_0 cannot be a singular point of any irreducible member of Λ, by Bézout theorem. We now analyze the presence of reducible members in Λ. Any reducible member that splits into the product of three lines must be the product $l_1 l_2 l_3$, for the collinearity relation we stated. Any reducible member that splits into the product of an irreducible conic and a line is Ql, since the line should pass through three distinct points and any choice of three collinear points different from p_2, p_4, p_7 determines a cubic split into three lines.

We denote by $P_0, \ldots, P_6$ the (-1)-curves above $p_0, \ldots, p_6$.
We will analyze the two reducible fibers separately. Let P, Q be two elements in the Mordell-Weil group of the rational elliptic surface given by the blow-up of $\mathbb{P}^2$ at the base points of Λ. Using the Contribution Table in section 1.9.2, we find that, if P meets the component Θ_i and Q meets the component Θ_j with $i \leq j$, the contribution of the I_4 fiber to $\langle P, Q \rangle$ is given by:

$$\mathrm{contr}(P, Q) = \frac{i(4-j)}{4}.$$

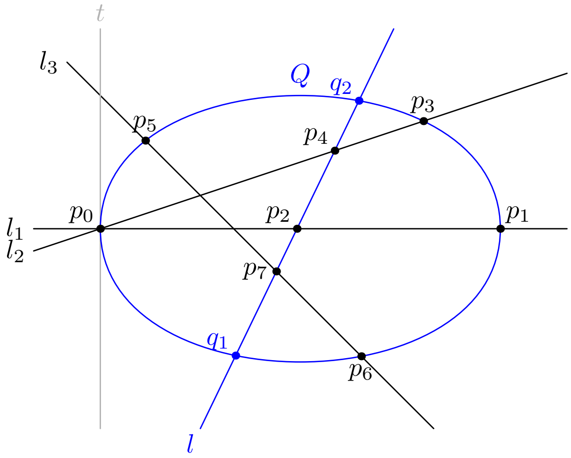


FIGURE 2.7. Configuration of the base points of a pencil of cubics inducing a rational elliptic surface with a fiber of type I_4 and a fiber with two components.

We set P_0 as the zero section. Using the technique described in section B.1 we find that, since p_0, p_1 and p_2 lie in the same line, then P_1 and P_2 intersect either Θ_1 or Θ_3. Suppose they intersect Θ_1. Then, for the same reason, P_3 and P_4 must intersect the component Θ_3 and all the other P_i's intersect the fiber at the component Θ_2. We get the following matrix:

$$(\mathrm{contr}_4(P_i, P_j))_{i,j} = \begin{pmatrix} 3/4 & 3/4 & 1/4 & 1/4 & 1/2 & 1/2 & 1/2 \\ 3/4 & 3/4 & 1/4 & 1/4 & 1/2 & 1/2 & 1/2 \\ 1/4 & 1/4 & 3/4 & 3/4 & 1/2 & 1/2 & 1/2 \\ 1/4 & 1/4 & 3/4 & 3/4 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1 & 1 & 1 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1 & 1 & 1 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1 & 1 & 1 \end{pmatrix}$$

For the fiber with two components we will proceed similarly. Using the Contribution Table in section 1.9.2, we find that, if P meets the component Θ_i and Q meets the component Θ_j with $i \leq j$, the contribution of the fiber to $\langle P, Q\rangle$ is given by:

$$\mathrm{contr}(P, Q) = \frac{i(2-j)}{2} = \begin{cases} 0 & \text{if} \quad i = 0 \\ 1/2 & \text{if} \quad i = j = 1. \end{cases}$$

Using the technique described in section B.1, since p_2, p_4 and p_7 are in the same line, and they are not in the conic where p_0 is, then P_2, P_4 and P_7 must intersect the component Θ_1. All the other P_i's intersect the zero component. We have the following matrix:

$$(\mathrm{contr}_2(P_i, P_j))_{i,j} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

This allows us to compute the height pairing of the points $P_1, \dots, P_6$.
We will consider the matrix given by the height pairing of P_4, P_5, P_6 and P_7. The height matrix is then the following:

$$A_{4,2} = \begin{pmatrix} 3/4 & 1/2 & 1/2 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}.$$

According to theorem 2.1.1, the Mordell-Weil lattice of the induced surface is isomorphic to $A_3^* \oplus A_1^*$; in particular it has determinant equal to $1/8$. Since the determinant of the matrix $A_{4,2}$ is equal to $1/8$, the elements P_4, P_5, P_6 and P_7 generate the full Mordell-Weil group of the rational elliptic surface.

THEOREM 2.4.2. *Let $\mathscr{E}$ be a rational elliptic surface with Mordell-Weil rank four and $MW(\mathscr{E}) \cong A_3^* \oplus A_1^*$. Then $\mathscr{E}$ arises from a linear pencil of cubic curves on as in construction* $(3)+(2')$.

PROOF. We must show that all the possible constructions of a rational elliptic surface of rank four with a fiber of type I_4 and a fiber with two components are equivalent to $(3)+(2')$.

For all the constructions where the fiber with two components is induced by an irreducible fiber, we can reduce to [Sal09], theorem 3.2, using the procedure described in section B.6. In fact, the configurations of the base points we have to deal with are similar to the ones treated in that theorem: the only difference is that in our cases there is a base point p with an infinitely near point, where p corresponds to the singular point of the member inducing the fiber with two components. This point p lies smoothly on the cubic inducing the I_4 fiber. Moreover, the equivalences we are looking for are given applying the Cremona transformations mentioned in that paper, being careful that the point p is not involved in the choices of the lines to contract.

All the equivalences will be proven as summarized in the following diagram (as we said, the first column of equivalences is granted):

$$\begin{tikzcd}
(1)+(1') \arrow[r, Rightarrow, "(i)"] \arrow[d, Leftrightarrow] & (1)+(2') \arrow[d, Rightarrow, "(ii)"] \\
(2)(b)+(1') \arrow[d, Leftrightarrow] & (2)(b)+(2') \arrow[d, Leftrightarrow, "(iii)"] \arrow[dd, Rightarrow, bend left, "(iv)"] \\
(2)(a)+(1') \arrow[d, Leftrightarrow] & (2)(a)+(2') \\
(3)+(1') & (3)+(2') \arrow[l, Rightarrow, "(v)"']
\end{tikzcd}$$

We now prove implication (i). Suppose we are working in the settings of construction $(1)+(1')$. Let $p_0, \dots, p_4$ be the base points of Λ and let $l_{i,j}$ be the line passing through p_i and p_j. Let p_0 be the base point of Λ with multiplicity 4 and let p_1 be the base point of Λ with multiplicity 2. Let p_2, p_3 be two other base points. Let $\mathscr{E}$ be the rational elliptic surface induced from Λ. Let P_i be the (-1)-curve above p_i in $\mathscr{E}$. It is possible to obtain a linear pencil of cubics as in $(1)+(2')$ contracting all the exceptional curves above p_0 (4 in total), the strict transforms of $l_{1,2}$, $l_{1,3}$ and $l_{2,3}$, and the (-1)-curves P_i, with $i \neq 2, 3$.

We now prove implication (ii). Suppose we are working in the settings of construction $(1)+(2')$. Let $p_0, \dots, p_5$ be the base points of Λ and let $l_{i,j}$ be the line passing through p_i and p_j. Let p_0 be the base point of Λ with multiplicity 4 and let p_1, p_2 be two base points on the conic belonging to the reducible member of Λ inducing the fiber with 2 components. Let $\mathscr{E}$ be the rational elliptic surface induced from Λ. Let P_i be the (-1)-curve above p_i in $\mathscr{E}$. It is possible to obtain a linear pencil of cubics as in $(2)+(2')$ contracting three of the four exceptional curves above p_0, the strict transforms of $l_{0,1}$ and of $l_{0,2}$ and each exceptional curve P_i with $i \neq 1, 2$.

We now prove the equivalence (iii). Suppose we are working in the settings of construction $(2)(b)+(2')$. Let $p_0, \dots, p_6$ be the base points of Λ and let $l_{i,j}$ be the line passing through p_i and p_j. Let p_0 be the base point of Λ with multiplicity 3. We suppose that the conics belonging to the reducible members of Λ are tangent to each other at p_0. Let p_1 be the base point of Λ realized as the intersection point of the line composing the cubic inducing the fiber of type I_4 and the conic composing the cubic inducing the fiber with two components. Let p_2 be a base point belonging to both those conics. Let $\mathscr{E}$ be the rational elliptic surface induced from Λ. Let P_i be the (-1)-curve above p_i in $\mathscr{E}$. It is possible to obtain a linear pencil of cubics as in $(2)(a)+(2')$ contracting two of the three exceptional curves above p_0, the strict transforms of $l_{0,2}$ and of $l_{1,2}$ and each exceptional curve P_i with $i \neq 1, 2$.
Vice versa, suppose we are working in the settings of construction $(2)(a)+(2')$. Let $p_0, \dots, p_6$ be the base points of Λ and let $l_{i,j}$ be the line passing through p_i and p_j. Let p_0 and p_1 be the base points of Λ with multiplicity 2 and let p_2 be the base point of Λ collinear with p_0 and p_2. Let p_3 and p_4 be two base points belonging to both the conics composing the reducible members of Λ. Let $\mathscr{E}$ be the rational elliptic surface induced from Λ. Let P_i be the (-1)-curve above p_i in $\mathscr{E}$. It is possible to obtain a linear pencil of cubics as in $(2)(b)+(2')$ contracting all the exceptional curves above p_0 (2 in total), the strict transforms of $l_{1,2}$, of $l_{1,3}$, of $l_{1,4}$, of $l_{2,3}$, and of $l_{2,4}$ and the (two) remaining exceptional curves above p_i with $i > 4$.

We now prove implication (iv). Suppose we are working in the settings of construction $(2)(b)+(2')$. Let $p_0, \dots, p_6$ be the base points of Λ and let $l_{i,j}$ be the line passing through p_i and p_j. Let p_0 be the base point of Λ with multiplicity 3. We suppose that the conics belonging to the reducible members of Λ have a common tangent at p_0. Let p_1 be a base point belonging to both the conics composing the reducible members of Λ. Let p_2 be one of the two points of intersection between the line composing the cubic inducing the fiber with two components and the conic composing the cubic inducing the fiber of type I_4. Let $\mathscr{E}$ be the rational elliptic surface induced from Λ. Let P_i be the (-1)-curve above p_i in $\mathscr{E}$. It is possible to obtain a linear pencil of cubics as in $(3)+(2')$ contracting two of the three exceptional curves above p_0, the strict transforms of $l_{0,1}$, of $l_{0,2}$, and of $l_{1,2}$ and all the exceptional curves P_i with $i > 2$.

We now prove implication (v). Suppose we are working in the settings of construction $(3)+(2')$. We use exactly the same notation as the one used in that construction. Let $l_{i,j}$ be the line passing through p_i and p_j. Let $\mathscr{E}$ be the rational elliptic surface induced from Λ. Let P_i be the (-1)-curve above p_i in $\mathscr{E}$. It is possible to obtain a linear pencil of cubics as in $(3)+(1')$ contracting all the exceptional curves above p_0 (2 in total), all the curves P_1, P_3, P_5, P_7 and the strict transforms of $l_{2,4}$, of $l_{2,6}$, and of $l_{4,6}$. □

2.4.3. Two Fibers with three Components. According to section B.4, in order to have two fibers with three components, the pencil of cubics must have two members of one of the following forms:

(1) A rational irreducible cubic such that the singular point is a base point with multiplicity 3,

(2) A reducible cubic, split into a line and an irreducible conic, such that only one of the intersection points between them is a base point and has multiplicity 2,
(3) A reducible cubic, split into 3 lines, such that no intersection points are base points.

2.4.3.1. *Construction.* We will construct an elliptic surface of rank 4 from a linear pencil of cubics as in (3) + (3).

Let E be a smooth cubic curve. Let l_1 be a line meeting E at three distinct points p_0, p_1, p_2. Let l_2 be another line intersecting l_1 at a point different from any p_i. We require also that

- the line l_2 meets E at three distinct points p_3, p_4, p_5 (all different from the previous p_i's),
- any line $m_i = \overline{p_{i-1}p_{i+2}}$ $(i = 1, 2, 3)$ is not tangent to E,
- the intersection points between m_i and m_j $(i \neq j = 1, 2, 3)$ are three distinct points not on E, and
- the new points p_{i+5}'s given by the intersection between any m_i and E (different from the previous p_j's) are not collinear to any non-trivial combination of the p_j's $j < 6$.

There exists a line l_3 passing through p_6, p_7, p_8.

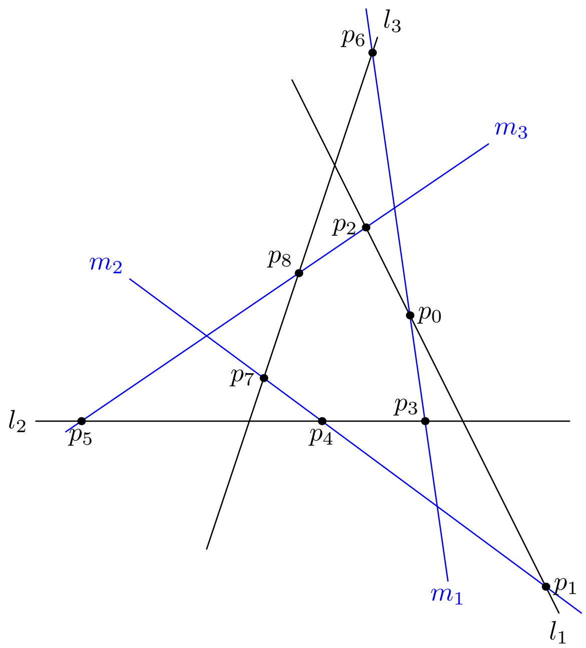


FIGURE 2.8. Configuration of the base points of a pencil of cubics inducing a rational elliptic surface with two fibers with three components each.

This holds if and only if $p_6 + p_7 + p_8 = q$ in the group law of E, where q is the third point of intersection between E and the tangent line to E at O. Using the collinearity relations:

$$p_0 + p_1 + p_2 = q, \qquad p_3 + p_4 + p_5 = q$$

and

$$p_0 + p_3 + p_6 = q, \qquad p_1 + p_4 + p_7 = q, \qquad p_2 + p_5 + p_8 = q,$$

we have

$$\begin{aligned} p_6 + p_7 + p_8 &= q - p_0 - p_3 + q - p_1 - p_4 + q - p_2 - p_5 \\ &= q + q - p_0 - p_1 - p_2 + q - p_3 - p_4 - p_5 \\ &= q + O + O = q. \end{aligned}$$

This choice grants that the pencil of cubics Λ, generated by $l_1l_2l_3$ and E contains $m_1m_2m_3$. We will show that $l_1l_2l_3$ and $m_1m_2m_3$ are the only reducible members in Λ. In fact, from the assumptions on the (non) collinearity of the p_i's the only lines passing through three of them are the one mentioned before. This shows that the only reducible members are $l_1l_2l_3$ and $m_1m_2m_3$. Moreover the base points of Λ cannot be the singular points of any irreducible cubic, by Bézout's theorem.

Notice that the reducible members of Λ correspond to fibers with three components in the rational elliptic surface given by the blow-up of $\mathbb{P}^2$ at the base points of Λ.

We denote by $P_0, \dots, P_8$ the (-1)-curves above $p_0, \dots, p_8$.
We denote by F_l the fiber induced by $l_1l_2l_3$ and by F_m the fiber induced by $m_1m_2m_3$. We will analyze the two reducible fibers separately. Let P, Q be two elements in the Mordell-Weil group of the rational elliptic surface given by the blow-up of $\mathbb{P}^2$ at the base points of Λ. Using the contribution table in section 1.9.2, we find that, if P meets the component Θ_i (of any of the two reducible fibers) and Q meets the component Θ_j with $i \le j$, the contribution of the fiber to $\langle P, Q\rangle$ is given by:

$$\mathrm{contr}(P,Q) = \frac{i(3-j)}{3}.$$

We set P_0 as the zero section. We are analyze the fiber F_l. Using the technique described in section B.1, since the points p_0, p_1 and p_2 lie in the same line, then P_1 and P_2 meet the zero component of the fiber, Θ_0; the points p_3, p_4 and p_5 lie in the same line, so P_3, P_4 and P_5 meet the same component, which can be either Θ_1 or Θ_2; without loss of generality, we fix it to be Θ_1; the points p_6, p_7 and p_8 lie in the same line, then P_6, P_7 and P_8 meet Θ_2. We get the following matrix:

$$(\mathrm{contr}_{F_l}(P_i, P_j))_{i,j} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 2/3 & 2/3 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 2/3 & 2/3 & 2/3 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 2/3 & 2/3 & 2/3 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 2/3 & 2/3 & 2/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 2/3 & 2/3 & 2/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 2/3 & 2/3 & 2/3 \end{pmatrix}$$

Now, we are analyzing the fiber F_m. The structure is similar to the previous one. We get the following matrix:

$$(\mathrm{contr}_{F_m}(P_i, P_j))_{i,j} = \begin{pmatrix} 2/3 & 1/3 & 0 & 2/3 & 1/3 & 0 & 2/3 & 1/3 \\ 1/3 & 2/3 & 0 & 1/3 & 2/3 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 1/3 & 0 & 2/3 & 1/3 & 0 & 2/3 & 1/3 \\ 1/3 & 2/3 & 0 & 1/3 & 2/3 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 1/3 & 0 & 2/3 & 1/3 & 0 & 2/3 & 1/3 \\ 1/3 & 2/3 & 0 & 1/3 & 2/3 & 0 & 1/3 & 2/3 \end{pmatrix}$$

We consider the matrix given by the height pairing of P_1, P_3, P_5 and P_6. The height matrix is the following:

$$A_{3,3} = \begin{pmatrix} 4/3 & 1 & 2/3 & 1 \\ 1 & 4/3 & 1/3 & 2/3 \\ 2/3 & 1/3 & 2/3 & 2/3 \\ 1/3 & 2/3 & 2/3 & 4/3 \end{pmatrix}.$$

According to theorem 2.1.1, the Mordell-Weil lattice of the induced surface is isomorphic to $A_2^{*\oplus 2}$; in particular it has determinant equal to $1/9$. Since the determinant of the matrix $A_{3,3}$ is equal to $1/9$, the elements P_1, P_3, P_5 and P_6 generate the full Mordell-Weil group of the rational elliptic surface.

THEOREM 2.4.4. *Let $\mathscr{E}$ be a rational elliptic surface with Mordell-Weil rank four and $MW(\mathscr{E}) \cong A_2^{*\oplus 2}$. Then $\mathscr{E}$ arises from a linear pencil of cubic curves as in construction* (3).

PROOF. We have to show that all the constructions are equivalent. Using the procedure described in section B.6, as for theorem 2.4.2, we can use an argument similar to the one used in [Fus06] to prove the equalities between the pencils having an irreducible member inducing a fiber with three components. All the equivalences will be proven as summarized in the following diagram:

$$\begin{array}{ccccc} (1)+(1) & & & & \\ \Updownarrow & & & & \\ (1)+(2) & \overset{(i)}{\Longleftrightarrow} & (2)+(2) & & \\ \Updownarrow & & & & \\ (1)+(3) & \overset{(ii)}{\Longleftrightarrow} & (2)+(3) & \overset{(iii)}{\Longleftrightarrow} & (3)+(3) \end{array}$$

We now show equivalence (i). Suppose we are in the setting of $(1)+(2)$. We denote by $p_0, \dots, p_5$ the base points of the pencil and by $l_{i,j}$ the line passing through p_i and p_j. Let $\mathscr{E}$ be the induced rational elliptic surface. We denote by P_i the (-1)-curve above p_i. Let p_0 be the base point of multiplicity 3. We suppose that the conic belonging to the cubic inducing one of the reducible fibers passes through p_0. Let p_1 be the base point with multiplicity 2. Let p_2 be a base point on the conic belonging to the cubic inducing one of the reducible fibers and let p_3 be a base point on the line belonging to the cubic inducing one of the reducible fibers. We obtain a linear pencil of cubics as in $(2)+(2)$ contracting two of the three exceptional curves above p_0, both the exceptional curves above p_1, the strict transforms of $l_{0,1}$, of $l_{0,2}$ and of $l_{1,2}$ and all the curves P_i with $i > 2$.
Vice versa, suppose we are working in the setting of $(2)+(2)$. We denote by $p_0, \dots, p_6$ the base points of the pencil and by $l_{i,j}$ the line passing through p_i and p_j. Let $\mathscr{E}$ be the induced rational elliptic surface. We denote by P_i the (-1)-curve above p_i. Both the reducible members of the pencil are split into a line and an irreducible conic. Let p_0, p_1, p_2 be the base points on one of the two lines, let p_2, p_3, p_4 be the three base points on the other line (p_2 is the intersection point between those lines) and let p_5, p_6 be the remaining base points (base points which are intersection points between the two conics different from p_0 and p_1). We suppose that p_0 and p_3 are the base points with multiplicity 2. We obtain a linear pencil of cubics as in $(1)+(2)$ contracting all the exceptional curves above p_0 and p_1 (4 in total), the strict transforms of $l_{1,4}$, of $l_{1,5}$, of $l_{4,5}$ and of $l_{0,1}$ and P_6.

We now prove equivalence (ii). Suppose we are in the setting of $(1)+(3)$. We denote by $p_0, \dots, p_6$ the base points of the pencil and by $l_{i,j}$ the line passing

through p_i and p_j. Let $\mathscr{E}$ be the induced rational elliptic surface. We denote by P_i the (-1)-curve above p_i. Let p_0 be the base point of multiplicity 3. Let p_1, p_2, p_3 three collinear base points and let p_4, p_5, p_6 be three other collinear base points. We can obtain a linear pencil of cubics as in $(2)+(3)$ contracting two of the three exceptional curves above p_0, the strict transforms of $l_{0,1}$, of $l_{0,4}$ and of $l_{1,4}$ and the curves P_2, P_3, P_5, P_6.
Vice versa, suppose we are in the setting of $(2)+(3)$. We denote by $p_0, \dots, p_7$ the base points of the pencil and by $l_{i,j}$ the line passing through p_i and p_j. Let $\mathscr{E}$ be the induced rational elliptic surface. We denote by P_i the (-1)-curve above p_i. Let p_0 be the base point of multiplicity 2. Let p_1 be the base point on the tangent at p_0 to the irreducible cubics of the pencil and let p_2, p_3 be the two base points collinear to p_0. We can obtain a linear pencil of cubics as in $(1)+(3)$ contracting both the exceptional curves above p_0, the strict transforms of $l_{1,2}$, of $l_{1,3}$ and of $l_{2,3}$ and all the curves P_i with $i > 2$.

We now show equivalence (iii). Suppose we are in the setting of $(2)+(3)$. We denote by $p_0, \dots, p_7$ the base points of the pencil and by $l_{i,j}$ the line passing through p_i and p_j. Let $\mathscr{E}$ be the induced rational elliptic surface. We denote by P_i the (-1)-curve above p_i. Let p_0 be the base point of multiplicity 2. Let p_1 be the base point on the tangent at p_0 to the irreducible cubics of the pencil and let p_2, p_3 be the two base points collinear to p_0. Let p_3, p_4, p_5 be three collinear base points and let p_2, p_6, p_7 be three collinear base points. We obtain a linear pencil of cubics as in $(3)+(3)$ contracting one of the two exceptional curves above p_0, the strict transforms of $l_{0,5}$, of $l_{0,7}$ and of $l_{5,7}$ and the curves P_1, P_2, P_3, P_4, P_6.
Vice versa, suppose we are working in the setting of $(3)+(3)$. We use the same notation used in that construction. We denote by $l_{i,j}$ the line passing through p_i and p_j. Let $\mathscr{E}$ be the induced rational elliptic surface. We denote by P_i the (-1)-curve above p_i. We can obtain a linear pencil of cubics as in $(2)+(3)$ contracting P_0, the strict transforms of $l_{2,3}$, of $l_{2,6}$ and of $l_{3,6}$ and the curves P_1, P_4, P_5, P_7, P_8. $\square$

2.5. Three Reducible Fibers

According to section 2.1 one reducible fiber has three components and the other two have two components. Appendix A tells us that the former fiber can be a of type IV or I_3 and latter ones of type I_2 or III.

2.5.1. A Fiber with three Components and two Fibers with two Components. In order to have a fiber with three components and two fibers with two, the pencil of cubics must contain one member of the following forms (see section B.4):

(1) A rational irreducible cubic such that the singular point is a base point with multiplicity 3,
(2) A reducible cubic, split into a line and an irreducible conic, such that only one of the intersection points between them is a base point and has multiplicity 2,
(3) A reducible cubic, split into 3 lines, such that no intersection points are base points;

and two members of the following forms (see section B.5):

(1′) A rational irreducible cubic such that the singular point is a base point with multiplicity 2,
(2′) A reducible cubic, split into an irreducible conic and a line, such that the intersection points between them are not base points.

2.5.1.1. *Construction.* We will construct an elliptic surface of rank 4 from a linear pencil of cubics as in $(3) + (2') + (2')$.

Let Q_a be an irreducible conic. Let $p_1, \ldots, p_6$ be six distinct points on Q_a such that the point p_0, given by the intersection of the line $l_b = \overline{p_5p_6}$ and $n_3 = \overline{p_2p_3}$ is not collinear with the points p_1 and p_4. We now define two lines $n_1 = \overline{p_6p_1}$ and $n_2 = \overline{p_4p_5}$. Take a point p_7 in n_2 and the point $p_8 = n_1 \cap \overline{p_0p_7}$. By Pascal's theorem there exists a (unique) conic Q_b passing through $p_1, \ldots, p_4, p_7, p_8$. This conic is irreducible if p_7 is a general point in n_2, that is $p_7 \notin \overline{p_3p_4}$. Now, call l_a the line $\overline{p_7p_8}$.

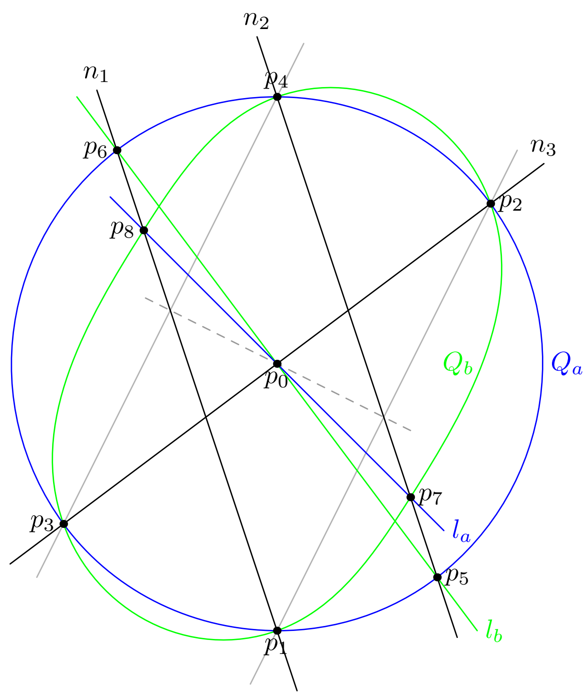


FIGURE 2.9. Configuration of the base points of a pencil of cubics inducing a rational elliptic surface with three reducible fibers.

Consider the pencil of cubics Λ, passing through $p_0, \ldots, p_8$. We show in section C.3 that Λ contains a smooth member. We now prove that the unique reducible members are $n_1n_2n_3$, Q_al_a and Q_bl_b. Suppose by contradiction that there exists a reducible member in Λ different from Q_al_a, Q_bl_b and $n_1n_2n_3$. It cannot split into three lines, since the only lines passing through point p_0 and two other base points are components of the above reducible members. This implies that this member is split into a line and an irreducible conic and p_0 is a point on that conic. This conic should pass through five other base points, such that no combination of those five points and p_0 leads to a reducible conic. This implies that the remaining three base points are not collinear, thus the line composing the member does not exists.

We will denote by $P_0, \ldots, P_8$ the (-1)-curves above $p_0, \ldots, p_8$.
We set P_0 as the zero section. The cubic $n_1n_2n_3$ determines a fiber of type I_3. Since there are nine distinct base points, the configuration of the these points easily determines the intersection between the exceptional curves and the components of the fiber, giving the following matrix, using the Contribution Table in section 1.9.2

in the case of a fiber with three components:

$$(\mathrm{contr}_3(P_i,P_j))_{i,j} = \begin{pmatrix} 2/3 & 0 & 0 & 1/3 & 1/3 & 2/3 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 2/3 & 2/3 & 1/3 & 2/3 & 1/3 \\ 1/3 & 0 & 0 & 2/3 & 2/3 & 1/3 & 2/3 & 1/3 \\ 2/3 & 0 & 0 & 1/3 & 1/3 & 2/3 & 1/3 & 2/3 \\ 1/3 & 0 & 0 & 2/3 & 2/3 & 1/3 & 2/3 & 1/3 \\ 2/3 & 0 & 0 & 1/3 & 1/3 & 2/3 & 1/3 & 2/3 \end{pmatrix}.$$

The cubic $Q_a l_a$ determines a fiber with two components. We can associate the following matrix:

$$(\mathrm{contr}_2^a(P_i,P_j))_{i,j} = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The cubic $Q_b l_b$ determines a fiber with two components. We can associate the following matrix:

$$(\mathrm{contr}_2^b(P_i,P_j))_{i,j} = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

We will consider the matrix given by the height pairing of P_1, P_3, P_5 and P_7. The height matrix is the following:

$$A_{3,2,2} = \begin{pmatrix} 1/3 & 0 & 1/6 & 1/6 \\ 0 & 1 & 1/2 & 1/2 \\ 1/6 & 1/2 & 5/6 & 1/3 \\ 1/6 & 1/2 & 1/3 & 5/6 \end{pmatrix}.$$

According to theorem 2.1.1, the Mordell-Weil lattice of the induced surface has determinant equal to $1/12$. Since the determinant of the matrix $A_{3,3}$ is the same, the elements P_1, P_3, P_5 and P_7 generate the full Mordell-Weil group of the rational elliptic surface.

THEOREM 2.5.2. *Let $\mathscr{E}$ be a rational elliptic surface with Mordell-Weil rank four with three reducible fibers. Then $\mathscr{E}$ arises from a linear pencil of cubic curves on as in construction* $(3)+(2')+(2')$.

PROOF. We have to prove that all the constructions of a rational elliptic surface with rank four and three reducible fibers are equivalent. As for theorem 2.4.2, we can use some results from [Sal09] and [Fus06], according to the procedure explained in section B.6: in fact, we can show that all the pencils having two irreducible members inducing the two fiber with two components are equivalent using the same reasoning used in [Fus06], in the context of a unique fiber with three component (we must be careful not to involve the singular points of the irreducible cubics inducing the fiber with two components in the choice of the lines to contract). For the same reason, we can reduce to [Sal09] for the equivalences between pencils with exactly one irreducible member inducing a fiber with two components and we can reduce to [Fus06] for the equivalences between pencils with an irreducible member inducing the fiber with three components.

All the equivalences will be proven as summarized in the following diagram (we denote by f^i the technique of [Fus06] in the context of i reducible fibers and by s^2 the technique of [Sal09] in the context of two reducible fibers):

$$\begin{tikzcd}
(1)+(1')+(1') \arrow[r, Leftrightarrow, "f^2"] \arrow[d, Leftrightarrow, "f^1"] & (1)+(1')+(2') \arrow[r, Leftrightarrow, "f^2"] \arrow[d, Leftrightarrow, "s^2"] & (1)+(2')+(2') \arrow[d, Leftrightarrow, "(i)"] \\
(2)+(1')+(1') \arrow[d, Leftrightarrow, "f^1"] & (2)+(1')+(2') \arrow[d, Leftrightarrow, "s^2"] & (2)+(2')+(2') \arrow[d, Leftrightarrow, "(ii)"] \\
(3)+(1')+(1') & (3)+(1')+(2') & (3)+(2')+(2')
\end{tikzcd}$$

We now prove equivalence (i). Suppose we are in the setting of $(1)+(2')+(2')$. We denote by $p_0, \dots, p_6$ the base points of the pencil and by $l_{i,j}$ the line passing through p_i and p_j. Let $\mathscr{E}$ be the induced rational elliptic surface. We denote by P_i the (-1)-curve above p_i. Let p_0 be the base point of multiplicity 3. Let C_i be the reducible cubic in the pencil, split into the conic Q_i and the line l_i. Let p_1 be the base point given by one of the intersection points of Q_1 and l_2 and let p_2 be the base point given by one of the intersection points of Q_2 and l_1. It is possible to obtain a pencil as in $(2)+(2')+(2')$ contracting two of the three exceptional curves above p_0, the strict transforms of $l_{0,1}$, of $l_{0,2}$ and of $l_{1,2}$ and P_3, P_4, P_5.
Vice versa, suppose we are working in the setting of $(2)+(2')+(2')$. We denote by $p_0, \dots, p_7$ the base points of the pencil and by $l_{i,j}$ the line passing through p_i and p_j. Let $\mathscr{E}$ be the induced rational elliptic surface. We denote by P_i the (-1)-curve above p_i. Let p_0 be the base point of multiplicity 2. Let p_1 be a base point given by the intersection of three conics each one of them belongs to a reducible member in the pencil. Let p_2 and p_3 be the two base points collinear with p_0. It is possible to obtain a linear pencil as in $(1)+(2')+(2')$ contracting both the exceptional curves above p_0, the strict transforms of $l_{1,2}$, of $l_{1,3}$ and of $l_{2,3}$ and the curves P_4, P_5, P_6, P_7.

We now prove equivalence (ii). Suppose we are working in the setting of $(2)+(2')+(2')$. We denote by $p_0, \dots, p_7$ the base points of the pencil and by $l_{i,j}$ the line passing through p_i and p_j. Let $\mathscr{E}$ be the induced rational elliptic surface. We denote by P_i the (-1)-curve above p_i. Let p_0 be the base point of multiplicity 2. Denote by C_i the reducible cubic of the pencil, split into a conic Q_i and a line l_i, inducing a fiber with two components. Let p_1 and p_2 be the two base points collinear with p_0. Let p_3 be a base point given by an intersection point of l_1 and Q_2 and let p_4 be a base point given by an intersection point of l_2 and Q_1. It i possible to obtain a linear pencil of cubics as in $(3)+(2')+(2')$ contracting only one exceptional curve above p_0, the strict transforms of $l_{0,3}$, of $l_{0,4}$ and of $l_{3,4}$ and P_1, P_2, P_5, P_6, P_7.
Vice versa, suppose we are in the setting of $(3)+(2')+(2')$. We use the same notation of that construction. We denote by $l_{i,j}$ the line passing through p_i and p_j and by P_i the (-1)-curve above p_i. It is possible to obtain a linear pencil of cubics as in $(2)+(2')+(2')$ contracting the strict transforms of $l_{3,6}$, of $l_{3,8}$ and of $l_{6,8}$ and $P_0, P_1, P_2, P_4, P_5, P_7$. $\square$

2.6. Four Reducible Fibers

According to section 2.1 all the reducible fibers have two components. From section A, each reducible fiber can be of type I_2 or III. We also recall that either the Mordell-Weil group has torsion, or it is torsion-free and both cases may happen (see theorem 2.1.1).

In order to have four fibers with two components, the pencil of cubics must contain four members of the following forms (see section B.5):

(1) A rational irreducible cubic such that the singular point is a base point with multiplicity 2,
(2) A reducible cubic, split into an irreducible conic and a line, such that the intersection points between them are not base points.

We first deal with the non-torsion case and a full section will be devoted for the torsion case.

2.6.1. The Non-Torsion Case. We are now trying to construct a rational elliptic surface of rank four with four reducible fibers without torsion.
We will build it as the blow-up of a linear pencil of cubics containing four reducible members split into a line and an irreducible conic.

First take two conics Q_a and Q_b, intersecting at four points p_1, p_2, p_3, p_4. Let l_d be the line passing through p_1 and p_2 and l_c be the line passing through p_3 and p_4. Let p_0 be the point of intersection between l_c and l_d. Now, take two points p_7 and p_8 on Q_b and two points p_5 and p_6 on Q_a satisfying the following requirements:

- the points p_0, p_7 and p_8 are collinear,
- the points p_0, p_5 and p_6 are collinear,
- the points p_1, p_2, p_5, p_6, p_7 and p_8 are on an irreducible conic Q_c and
- the points p_3, p_4, p_5, p_6, p_7 and p_8 are on an irreducible conic Q_d.

We will denote by l_a the line passing through p_7 and p_8 and by l_b the line passing through p_5 and p_6.

Consider the linear pencil of cubics Λ passing through $p_0, \dots, p_8$. We prove in section C.4 that Λ contains a smooth member. Now we analyze the presence of reducible members. The reducible members in Λ split into a line and a conic. Since the base points of Λ are nine distinct points, any line should pass through three base points. The only possibility for that line is to be one among $l_a, \dots, l_d$, since there cannot be other collinearity relations between the p_i's, otherwise at least one among $Q_a, \dots, Q_d$ would be a reducible conic. This shows that Λ has exactly $Q_a l_a, \dots, Q_d l_d$ as reducible members.

Let $P_0, \dots, P_8$ be the (-1)-curves above $p_0, \dots, p_8$. Set P_0 as the zero section. Every reducible member in Λ determines a fiber with two components. Using the same reasoning as in the previous constructions, we can deduce the contribution of each fiber to each pairing between the P_i's. The height matrix associated to $P_1, \dots, P_8$ is the following:

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Taking the submatrix given by the pairing of the exceptional curves above four base points such that any two of them are not on any line l_a, l_b, l_c or l_d, we get a matrix

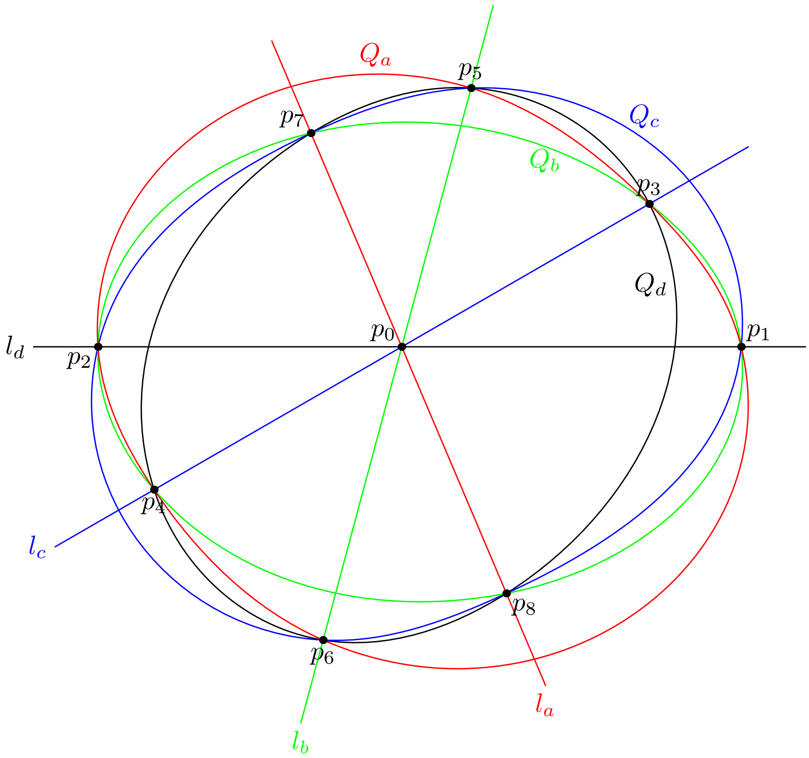


FIGURE 2.10. Configuration of the base points of a pencil of cubics inducing a rational elliptic surface with four reducible fibers.

of the following form:

$$\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}.$$

This matrix has determinant equal to $1/16$, which is exactly the determinant associated to the Mordell-Weil group of the induced rational elliptic surface. Thus the exceptional curves generate the full Mordell-Weil lattice of the surface. Moreover the possibility that the surface has torsion is excluded: in the case of a rational elliptic surface with rank four with torsion, the determinant of the height matrix of any four elements is equal to $1/4$ times a square integer. This happens because the determinant of the associated Mordell-Weil lattice is $1/4$.

2.7. The Torsion Case

2.7.1. Construction. In this section we will construct a linear pencil of cubics with four reducible members, split into a line and an irreducible conic such that the induced rational elliptic surface has torsion.

To improve readability we will use the notation $r \cap s$ for "r intersected with s" and $p \cup q$ for "the line passing through p and q".

First, take five points p_1, p_2, a, b and c. Let β be the line

$$\beta = ((p_2 \cup a) \cap (p_1 \cup c)) \cup ((p_1 \cup b) \cap (p_2 \cup c)).$$

Let α and δ be two lines defined by

$$\alpha = a \cup ((p_1 \cup c) \cap (p_2 \cup b)),$$
$$\delta = b \cup ((p_1 \cup a) \cap (p_2 \cup c)).$$

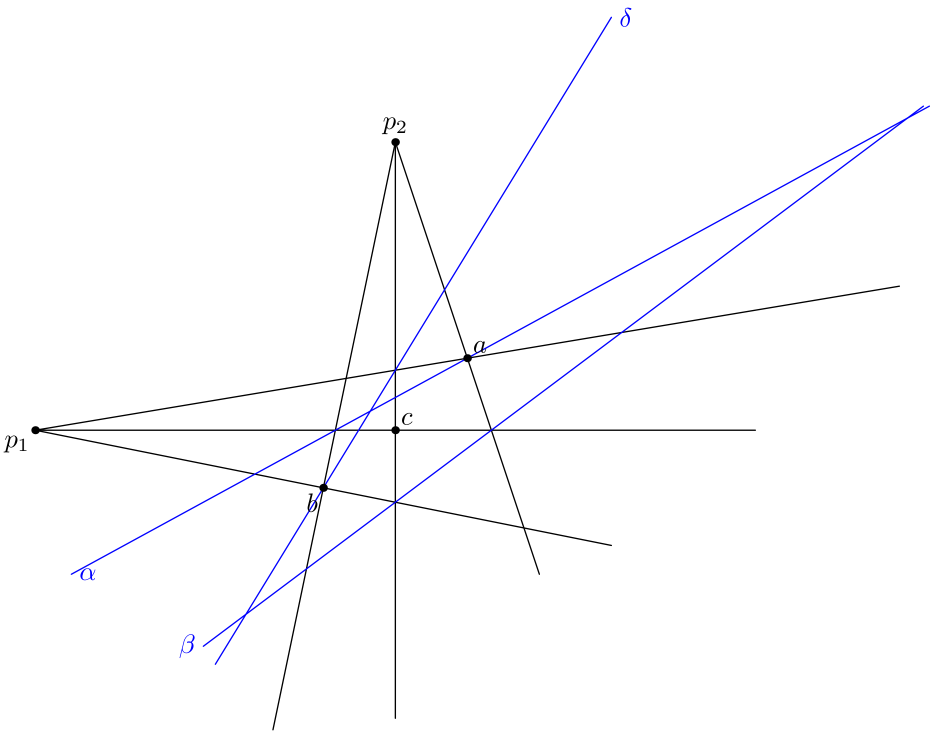


FIGURE 2.11. Construction of the Pascal's lines.

The blue lines, denoted by Greek letters, will be crucial in proving the presence of four conics passing through six points. In this step we will build the first two lines appearing as components of two of the reducible members of the pencil of cubics.

Let p_0 be the point

$$((\beta \cap \delta) \cup p_1) \cap ((\beta \cap \alpha) \cup p_2).$$

Let l_a be the line

$$((p_0 \cup p_1) \cap \alpha) \cup ((p_1 \cup a) \cap (p_2 \cup b))$$

and let l_b be the line

$$((p_0 \cup p_2) \cap \delta) \cup ((p_1 \cup a) \cap (p_2 \cup b)).$$

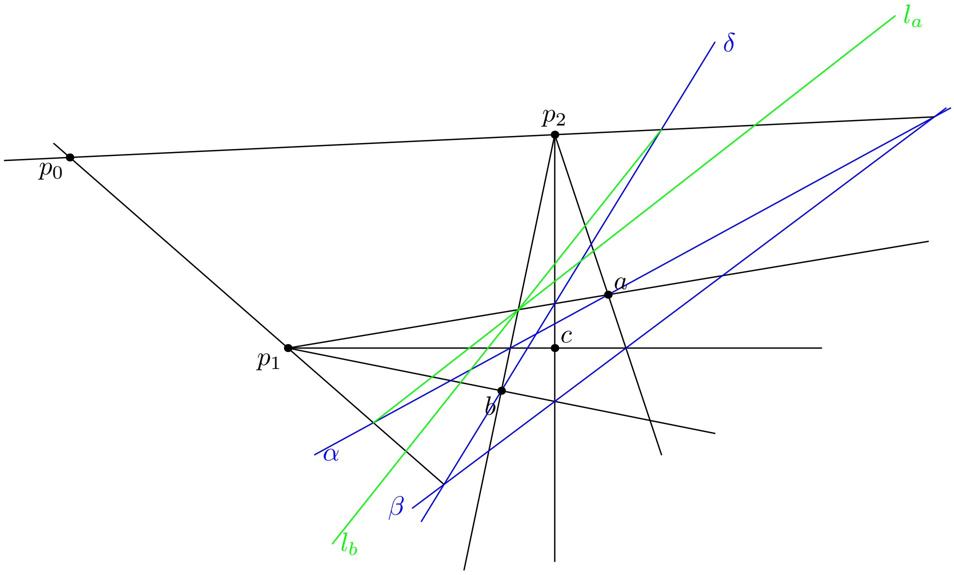


FIGURE 2.12. Setting the first collinear base points.

In this stage we will complete the set of base points. Let l_c be the line

$$(l_a \cap (p_1 \cup b)) \cup (l_b \cap (p_1 \cup c))$$

and let l_d be the line

$$(l_b \cap (p_2 \cup a)) \cup (l_a \cap (p_2 \cup c)).$$

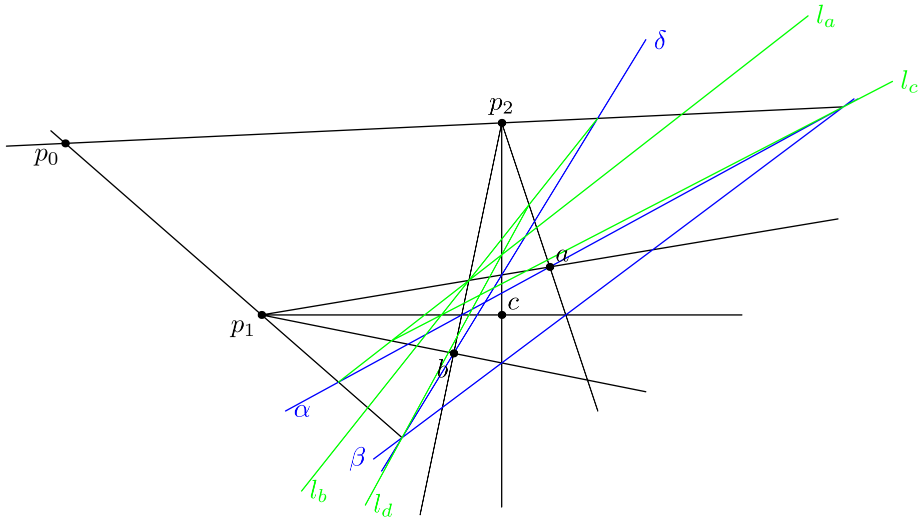


FIGURE 2.13. The final setup.

We label the intersection points of the l_i's as follows:

$$p_3 = l_a \cap l_b, \quad p_4 = l_a \cap l_d, \quad p_5 = l_b \cap l_c,$$
$$p_6 = l_c \cap l_d, \quad p_7 = l_a \cap l_c, \quad p_8 = l_b \cap l_d.$$

We fixed the position of nine points in $\mathbb{P}^2$. In section D.2 we prove that the p_i's are base points of a pencil of cubics with four reducible members split into a line and an irreducible conic. We check that the pencil contains at least a smooth member in section C.5.

2.7.2. The Mordell-Weil Group. We just showed that it is possible to choose nine distinct points $p_0, \ldots, p_8$ on $\mathbb{P}^2$ such that there are four lines $l_a, \ldots, l_d$ and four irreducible conics $Q_a, \ldots, Q_d$ with the following properties:

$$p_3, p_5, p_8 \in l_a, \qquad p_3, p_4, p_7 \in l_b, \qquad p_5, p_6, p_7 \in l_c, \qquad p_4, p_6, p_8 \in l_d,$$
$$p_0, p_1, p_2, p_4, p_6, p_7 \in Q_a, \qquad p_0, p_1, p_2, p_5, p_6, p_8 \in Q_b,$$
$$p_0, p_1, p_2, p_3, p_4, p_8 \in Q_c, \qquad p_0, p_1, p_2, p_3, p_5, p_7 \in Q_d,$$

any three other combination of the p_i's is not on a line and any other combination of six of the p_i's is not on an irreducible conic.

Consider the pencil Λ determined by $p_0, \ldots, p_8$. The unique reducible members are $Q_a l_a, Q_b l_b, Q_c l_c$ and $Q_d l_d$, since for every other conic passing through six of the p_i's the remaining three base points are not collinear.

We will denote by $P_0, \ldots, P_8$ the exceptional curves above $p_0, \ldots, p_8$. We set P_0 as the zero section. Every reducible member of Λ determines a fiber with 2 components. With the same method used in the other constructions, it is possible to get the intersection matrices for every reducible fibers and combine them in order to get the height matrix of the P_i's ($i \neq 0$). The height matrix is the following:

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1/2 & 1/2 & 1 & 1/2 & 1/2 \\ 1 & 1 & 1/2 & 1 & 1 & 1/2 & 1/2 & 1/2 \\ 1 & 1 & 1/2 & 1 & 1 & 1/2 & 1/2 & 1/2 \\ 1 & 1 & 1 & 1/2 & 1/2 & 1 & 1/2 & 1/2 \\ 1 & 1 & 1/2 & 1/2 & 1/2 & 1/2 & 1 & 1 \\ 1 & 1 & 1/2 & 1/2 & 1/2 & 1/2 & 1 & 1 \end{pmatrix}.$$

As for the other constructions, we would like to show that the exceptional curves above the p_i's generate the full Mordell-Weil group associated to the pencil of cubics. In our case the Mordell-Weil group is isomorphic to $D_4^* \oplus \mathbb{Z}/2\mathbb{Z}$ (see theorem 2.1.1). This means that in order to generate the Mordell-Weil group, we need to find four exceptional curves above the base points of the pencil that generate the Mordell-Weil lattice D_4^* (since the rank of the rational elliptic surface is four) and check that there is a linear combination of the exceptional curves that leads to a non-zero 2-torsion element.

For the latter part, one can check that $2(P_3 - P_6) = O$ (and $2(P_4 - P_5) = O$, $2(P_7 - P_8) = O$), as done in subsection C.5.1. For the first part, we choose any submatrix of A given by four independent elements and get one of the following matrices (the first one if we choose only one between P_1 and P_2 and three independent P_i's, $i > 2$; the second one if we choose both P_1 and P_2 and two independent P_i's, $i > 2$):

$$B_1 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1/2 & 1/2 \\ 1 & 1/2 & 1 & 1/2 \\ 1 & 1/2 & 1/2 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1/2 \\ 1 & 1 & 1/2 & 1 \end{pmatrix}.$$

In both cases the determinant is $1/4$. This shows that the elements we chose generate the Mordell-Weil lattice, since

$$\det(B_i) = \frac{1}{4} = \det(D_4^*),$$

and implies that the exceptional curves above the base points of the linear pencil of cubics generate the full Mordell-Weil group.

APPENDIX A

Fiber Configuration

In this section we will list all possible special fibers that can occur in a rational elliptic surface, together with the Kodaira symbol, the order and the number of irreducible components.

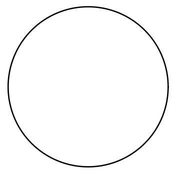
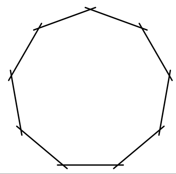
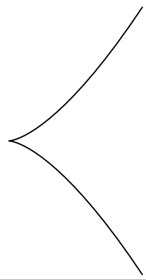
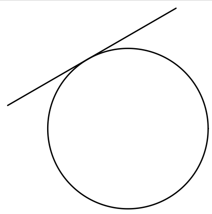
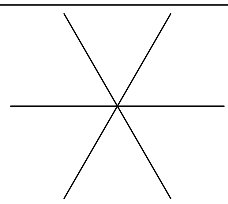
Kodaira Symbol	$\mathbf{ord_v(\Delta_v)}$	**# irreducible components**	**Drawing**
I_0	0	1	
I_n	n	n	
II	2	1	
III	3	2	
IV	4	3	

TABLE A.1. Fibers and properties 1/2.

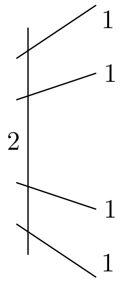
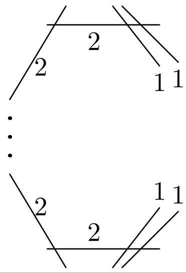
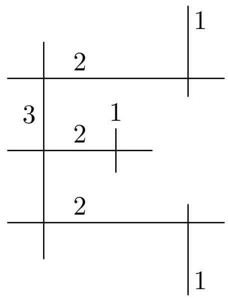
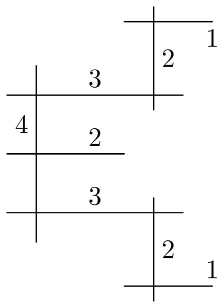
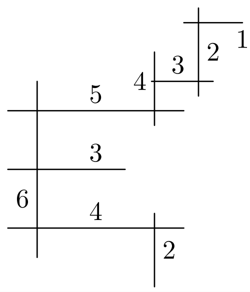
Kodaira Symbol	$\mathbf{ord_v(\Delta_v)}$	**# irreducible components**	**Drawing**
I_0^*	6	5	
I_n^*	$6+n$	$5+n$	
IV^*	8	7	
III^*	9	8	
II^*	10	9	

TABLE A.2. Fibers and properties 2/2.

APPENDIX B

Singular Cubics

B.1. Description of the Fibers

From section 1.7, each rational elliptic surface $\pi : \mathscr{E} \to \mathbb{P}^1$ is obtained as the blow-up of $\mathbb{P}^2$ at the base points of a linear pencil of cubics Λ. This correspondence yields a more specific relation: namely, for each fiber $\pi^{-1}(t)$ we can associate a specific member Λ_t whose image after the blow-up is exactly $\pi^{-1}(t)$.

First, define m_p as the multiplicity of p as a base point of the pencil; that is, $m_p = i$ means that p is a base point having $i-1$ infinitely-near base points. In order to obtain the fiber of π corresponding to the curve C, we do the following for each base point p of Λ:

- if $m_p > 0$, we blow-up the point; otherwise we quit.
- we set p' as the strict transform of p in the proper transform C' of C. the value of $m_{p'}$ is $m_p - 1$. The curve C' is the strict transform of C together with the exceptional curve E above p counted with multiplicity $m_C(p)-1$.
- we re-label p' as p and C' as C and start again the procedure.

The final shape of the fiber associated to the original curve C is determined by applying this procedure to each base point of Λ. Thus, the fiber is given by the strict transform of the original curve C plus all the (-2)-curves gained by the procedures.

B.1.1. Example. In order to make the reader familiar with this technique (that we use very often), we give an example of an I_5 fiber induced by a member in Λ split into three lines.

Suppose that the linear pencil Λ contains a member C, given by the product of three non-concurrent lines, and only one of the three intersection points between them is a base point a with $m_a = 3$ and all the other base points p have $m_p = 1$. Since $m_a = 3$, one of the lines composing C a belongs to is tangent at a to all the smooth cubics in Λ. Let l be this line. The shape of this curve is given in Figure B.1, by Bézout's theorem (numbers correspond to multiplicities as base points). We are going to first analyze the blow-up at the point a, in Figure B.2. After the first blow-up, a' will be the intersection point between (the image of) l and the exceptional curve E_1 above a (E_1 is a (-1)-curve at this stage). The curve C' is given by the strict transform of C plus the exceptional curve E_1 above a, counted with multiplicity 1.
After re-labeling and blowing-up the second time, a' is a general point on the exceptional curve E_2 above a (E_2 is a (-1)-curve); (the image of) the curve E_1 is now a (-2)-curve (denoted by E_1, again); the curve C' is the strict transform of C plus the exceptional curve E_2 above a, counted with multiplicity 1.

Now, all (the images via the blow-ups of) the base points have multiplicity 1, so their blow-up do not affect the shape of the fiber. The blow-up of each point p among these points will just give a (-1)-curve intersecting (the strict transform of) the component of the fiber p belonged to. The final configuration is given in Figure B.3.

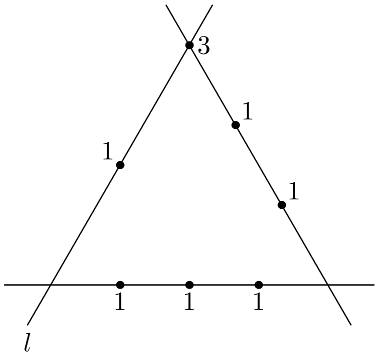


FIGURE B.1. Position of the base points on the curve C.

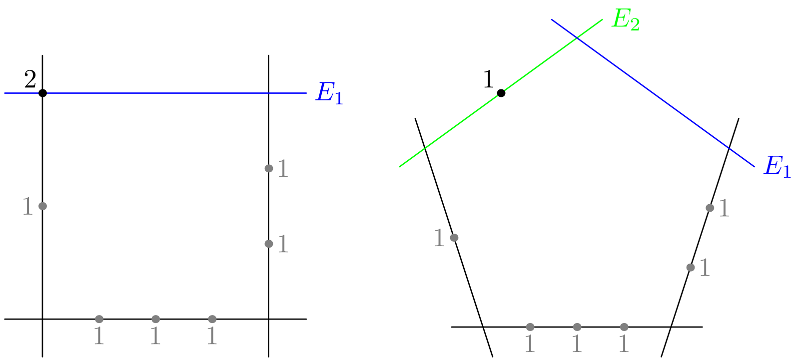


FIGURE B.2. Position of the base points on the blow-ups at the point a.

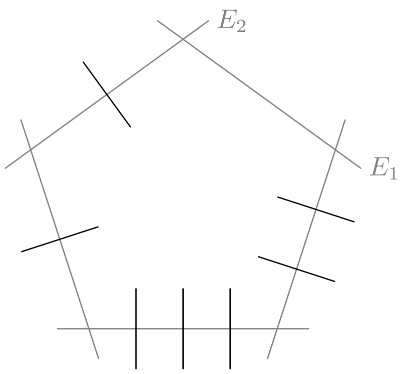


FIGURE B.3. Position of the (-1)-curves over the base points on the fiber.

B.2. A Fiber with Five Components

In the following table we summarize the possible cubics in Λ that can be the image of a fiber with 5 components; from section A, we know that such fiber can be a fiber of type I_5 or I_0^*. The table shows the whole procedure, so the actual cubics in $\mathbb{P}^2$ are divided from the curves that are not cubics, in order to immediately see the possible members in Λ.

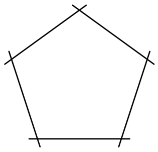
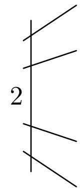
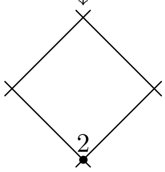
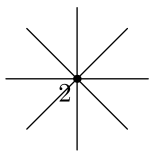
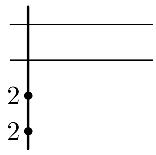
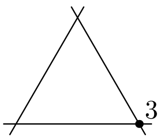
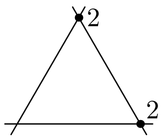
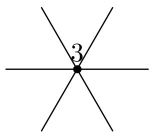
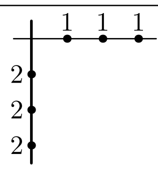
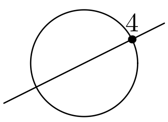
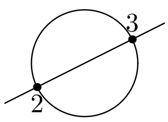
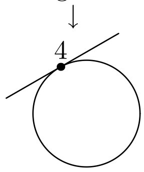
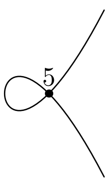
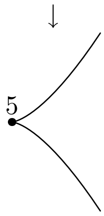
I_5	I_0^*
	
A fiber of type I_5. ↓	A fiber of type I_0^*. ↓
	 
4 non concurrent lines; one intersection point is double. ↓	4 concurrent lines or a double line and two non concurrent lines (the latter after 2 contractions). ↓
 	 
3 non concurrent lines. ↓	3 concurrent lines or a double line and a single line. ↓
 	
An irreducible conic and a non tangent line. ↓	An irreducible conic and a tangent line. ↓
	
A node.	A cusp.

TABLE B.1. Fibers with five components.

B.3. A Fiber with Four Components

In the following table we summarize the possible cubics in Λ that can be the image of a fiber with 4 components; from section A, we know that such fiber can only be a fiber of type I_4. The table shows the whole procedure, so the actual cubics in $\mathbb{P}^2$ are divided from the curves that are not cubics, in order to immediately see the possible members in Λ.

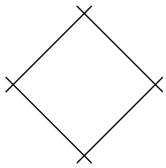
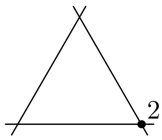
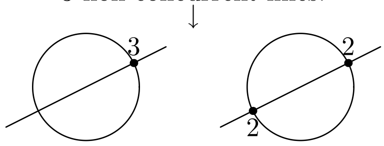
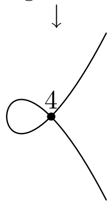
I_4
 A fiber of type I_4. $\downarrow$
 3 non concurrent lines. $\downarrow$  An irreducible conic and a non tangent line. $\downarrow$  A node.

Table B.2. A fiber with four components.

B.4. A Fiber with Three Components

In the following table we summarize the possible cubics in Λ that can be the image of a fiber with 3 components; from section A, we know that such fiber can be a fiber of type I_3 or IV.

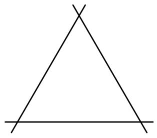
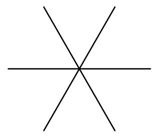
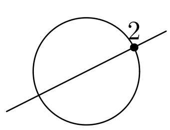
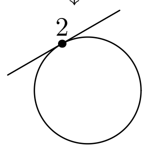
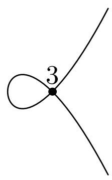
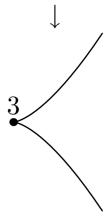
I_3	IV
 A fiber of type I_3. $\downarrow$	 A fiber of type IV. $\downarrow$
 An irreducible conic and a non tangent line. $\downarrow$	 An irreducible conic and a tangent line. $\downarrow$
 A node.	 A cusp.

TABLE B.3. Fibers with three components.

B.5. A Fiber with Two Components

In the following table we summarize the possible cubics in Λ that can be the image of a fiber with 2 components; from section A, we know that such fiber can be a fiber of type I_2 or III.

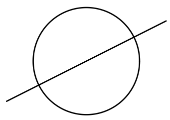
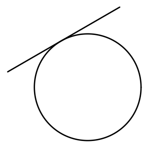
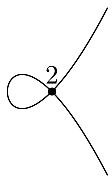
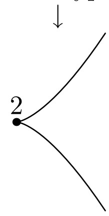
I_2	III
	
A fiber of type I_2.	A fiber of type III.
$\downarrow$	$\downarrow$
	
A node.	A cusp.

TABLE B.4. Fibers with two components.

B.6. Reduction to Other Papers

In this section we give a brief description of how to reduce a proof of an equivalence to another equivalence between pencils of higher rank.

We say that two pencils are equivalent if they are birational to each other.

In the papers dealing with higher rank rational elliptic surfaces, as well as in this thesis, equivalences between pencils are proven using Cremona transformations. The goal is to separate infinitely near base points of the pencil and/or to merge distinct base points of the pencil. In both cases, we can reach the result changing the choice of the (-1)-curves to contract on the induced rational elliptic surface.

If we want to separate one infinitely near base point from a base point p having $m_p - 1$ infinitely near base points, we contract all the exceptional curves above p except one (there are m_p exceptional curves; we contract $m_p - 1$ of them) and then we contract the strict transforms of the lines $\overline{pa}$, $\overline{pb}$ and $\overline{ab}$, where a and b are two other distinct base points having both no infinitely near base points. We do not contract the (-1)-curves above a and b, neither the last exceptional curve above p. Every exceptional curve above the other base points is contracted as before.

If in the original pencil there is a base point c different not lying in any line $\overline{pa}$, $\overline{pb}$ or $\overline{ab}$ that has $m_c - 1$ infinitely near base points, there is a base point c' having $m_c - 1$ infinitely near points also in the resulting pencil.

This procedure can be used in order to save some work: suppose that two linear pencils of cubics Λ_a and Λ_b were proven to be equivalent. Suppose that $a_1, \ldots, a_t$ are the base points of Λ_a and $b_1, \ldots, b_t$ are the base points of Λ_b and that in the proof of the equivalence between Λ_a and Λ_b the birational maps (described above) involve just the choice of $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ $(3 < k < t-1)$. That proof holds also for the equivalence of $\Lambda_{a'}$ and $\Lambda_{b'}$ having $a_1, \ldots, a_k, a'_{k+1}, \ldots, a'_t$ and $b_1, \ldots, b_k, b'_{k+1}, \ldots, b'_t$ respectively as base points, with the only restrictions that the points $a'_{k+1}, \ldots, a'_t$ have the same configuration of $b'_{k+1}, \ldots, b'_t$.

This fact allows us to use the proofs in [**Sal09**] and [**Fus06**] whenever the conditions on the pencils are met. In the reductions we make, we often use the equivalence between Λ_a and Λ_b where the birational maps involve only the points $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ ($k < 8$) in order to prove that the pencils $\Lambda_{a'}$ and $\Lambda_{b'}$ are equivalent, where $a'_i = a_i$ and $b'_i = b_i$ for all $i = 1, \ldots, t$ and $a'_t = a'_{t+1}$ (and $b'_t = b'_{t+a}$, consequently).

APPENDIX C

Pencils of Cubics from Reducible Generators

In this chapter we will prove that our constructions of linear pencils of cubics with reducible generators actually produce smooth pencils of elliptic curves.
We need to prove that the pencils we construct have at least a smooth generator. When we constructed rational elliptic surfaces with only one reducible fiber, we built pencils with a smooth generator, so in these cases there is nothing to prove. This is true also in the case of rational elliptic surfaces with two reducible fibers with three components. The only cases left are the ones with two reducible fibers with four and two components, the one with three reducible fibers and the cases with four reducible fibers. In all the cases we will use the same procedure.

C.1. Technique

Let Λ be a linear pencil of cubics. Denote its base points by $p_0, \ldots, p_8$. To show that there exists a smooth cubic passing through the nine points p_i's, we will take an elliptic curve (E, O) over eight of them (this is possible in all constructions) and then, using the group structure of E, we prove that the ninth point is on E. This shows that the nine points we had at the beginning determine a smooth pencil of cubics.

First we should check that there exists an elliptic curve through eight points among the base points of a linear pencil coming from the constructions we want to deal with. This is always the case, since there are not more than three points on a line and not more than six points on a conic; Bézout's theorem grants the existence of a pencil of smooth cubics through those eight points.

C.1.1. The Group Law. The group law of a plane elliptic curve (E, O) can be explained geometrically. From now on we denote by q the third point of intersection between E and the tangent to E at O. Three collinear points a, b, c add up to q.

In particular, for an elliptic curve in Weierstrass form, O is an inflection point and we have that $q = O$: this leads to the usual group law ("three collinear points add up to zero").

The following Lemma describes the only further property of the group law that we need for specialized configurations of base points.

Lemma C.1.2. *Let (E, O) be a plane elliptic curve. Let q be the third point of intersection between E and the tangent to E at O. Let Q be a conic.*

Then, the six intersection points between E and Q add up to $2q$.

Proof. We know that in the divisor group of an elliptic curve the sum of three collinear points a, b, c is linearly equivalent to the sum of any three collinear points. In particular (taking the tangent line at O) we have the following equality in the Picard group:

$$a + b + c = 2 \cdot O + q = q.$$

Consider now the relations given by the hyperplane sections of degree 2; that is, the relations between sums of divisors of intersection between the elliptic curve and conics. Let Q be any conic. Let $p_1, \ldots, p_6$ be the intersection points between Q and the elliptic curve. Consider now the conic given by twice the tangent line to the elliptic curve at O. Using linear equivalence again, we obtain the result:

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 2 \cdot (2 \cdot O + q) = 2 \cdot q.$$

□

C.1.3. Notation. In the next sections, we need to write some relations. We will write them as equations (in the group determined by E) and sometimes we will put an extra label after the equation, to explain from which membership it is determined. For example, if l is a line,

$$a + b + c = q \quad (l)$$

means that a, b, c are the three points of intersection between E and the line l; if Q is a conic

$$a + b + c + d + e + f = 2q \quad (Q)$$

means that a, b, c, d, e, f are the six points of intersection between E and the conic Q.

We will also add or subtract such equations; we will keep track of these operations on the labels: for example,

$$d + e + f = q \quad (Q) - (l)$$

means that subtracting (l) to (Q) we get that the points d, e, f add up to q; this also implies that they are collinear (the conic Q is reducible).

C.2. A Pencil with two Reducible Members

In this section we prove that the construction in subsection 2.4.1 leads to a pencil containing a smooth cubic. We will use the same notation of that construction.

Suppose we fixed all the base points p_i's as described in subsection 2.4.1. We want to show that all the cubics passing through eight p_i's pass also through the ninth one.

Suppose that E is any smooth cubic passing through $p_0, \ldots, p_6$ with tangent t at p_0. We want to show that E passes through p_7. We denote by p_n the point of intersection between E and l_c that is not p_5 nor p_6 and p_l the point of intersection between E and l that is not p_2 nor p_4. Restating our problem, we need to show that $p_n = p_l$. We will use the following relations:

$$\begin{aligned} p_0 + p_1 + p_2 &= q \quad (l_a), \\ p_0 + p_3 + p_4 &= q \quad (l_b), \\ p_5 + p_6 + p_n &= q \quad (l_c), \end{aligned} \qquad \begin{aligned} 2p_0 + p_1 + p_3 + p_5 + p_6 &= 2q \quad (Q), \\ p_2 + p_4 + p_l &= q \quad (l). \end{aligned}$$

Now, subtracting (Q) to (l_c) and adding (l_a) and (l_b) we get that

$$p_2 + p_4 + p_n = q.$$

This implies that $p_n = p_l$ and the proof is concluded.

C.3. A Pencil with three Reducible Members

In this section we prove that the construction in section 2.5 leads to a pencil containing a smooth cubic. We will use the same notation of that construction.

Suppose we fixed all the base points p_i's as described in section 2.5. We want to show that all the cubics passing through eight p_i's pass also through the ninth one.

Suppose that E is any smooth cubic passing through $p_1, \ldots, p_8$. There exist three points p_n, p_a, p_b such that

$$n_3 \cap E = p_2 + p_3 + p_n$$
$$l_a \cap E = p_7 + p_8 + p_a$$
$$l_b \cap E = p_5 + p_6 + p_b.$$

We need to show that $p_n = p_a = p_b$ $(= p_0)$.
We will use the group law on E. Denote by q the third point of intersection between E and the tangent to E at 0. Then, according to the notation explained in section C.1.3, we can rewrite the above formulas as

$$p_2 + p_3 + p_n = q \quad (n_3), \qquad p_7 + p_8 + p_a = q \quad (l_a), \qquad p_5 + p_6 + p_b = q \quad (l_b)$$

and all the previous relations (points on lines and points on conics) as

$$p_1 + p_6 + p_8 = q \quad (n_1), \qquad p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 2q \quad (Q_a),$$
$$p_4 + p_5 + p_7 = q \quad (n_2), \qquad p_1 + p_2 + p_3 + p_4 + p_7 + p_8 = 2q \quad (Q_b).$$

Subtracting (l_b) from (Q_a), we get that

$$p_1 + p_2 + p_3 + p_4 - p_b = q$$

and subtracting (l_a) from (Q_b), we get that

$$p_1 + p_2 + p_3 + p_4 - p_a = q.$$

Thus $p_a = p_b$. We still have to show that p_n is the same point as p_a and p_b. This can be proved combining some previous relations; we start from $(n_3) + (n_1) + (n_2)$:

$$p_n + p_2 + p_3 + p_1 + p_6 + p_8 + p_4 + p_5 + p_7 = 3q$$

and subtract (Q_a)

$$p_n + p_7 + p_8 = q.$$

This implies $p_n = p_a$, subtracting (l_a). We have shown that p_n, p_a and p_b are actually the same point, so they coincide with p_0. This shows that the cubic E passes through the ninth base point of Λ and the proof is concluded.

C.4. A Pencil with four Reducible Members: the non-Torsion Case

In this section we prove that the construction in subsection 2.6.1 leads to a pencil containing a smooth cubic. We will use the same notation of that construction.

Suppose we fixed the base points $p_0, \ldots, p_8$. We want to show that any cubic passing through eight of them passes through the ninth one. Let E be a smooth cubic passing through $p_0, \ldots, p_7$. There exist four points $x_a, \ldots, x_d$ such that (according to the notation explained in section C.1.3):

$$p_0 + p_7 + x_a = q \quad (l_a),$$
$$p_1 + p_2 + p_3 + p_4 + p_7 + x_b = 2q \quad (Q_b),$$
$$p_1 + p_2 + p_5 + p_6 + p_7 + x_c = 2q \quad (Q_c),$$
$$p_3 + p_4 + p_5 + p_6 + p_7 + x_d = 2q \quad (Q_d).$$

Moreover:

$$p_0 + p_1 + p_2 = q \quad (l_d),$$
$$p_0 + p_3 + p_4 = q \quad (l_c),$$
$$p_0 + p_5 + p_6 = q \quad (l_b),$$
$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 2q \quad (Q_a).$$

Adding $(l_a), (l_b)$ and (l_c) and subtracting (Q_a), we get that $3p_0 = q$. Plugging this information in (Q_b), (Q_c) and (Q_d) we get that $x_a = \cdots = x_d$. This concludes the proof.

C.5. A Pencil with four Reducible Members: the Torsion Case

In this section we prove that the construction in section 2.7.1 leads to a pencil containing a smooth cubic. We will use the same notation of that construction.

Suppose we fixed all the base points p_i's as described in sections 2.7.1 and 2.7. We want to show that all the cubics passing through eight p_i's pass also through the ninth one.

Suppose that E is any smooth cubic passing through $p_1, \ldots, p_8$. There exist four points x_a, x_b, x_c and x_d such that (according to the notation explained in section C.1.3):

$$\begin{aligned} x_a + p_1 + p_2 + p_5 + p_6 + p_8 &= 2q \quad (Q_a), \\ x_b + p_1 + p_2 + p_4 + p_6 + p_7 &= 2q \quad (Q_b), \\ x_c + p_1 + p_2 + p_3 + p_4 + p_8 &= 2q \quad (Q_c), \\ x_d + p_1 + p_2 + p_3 + p_5 + p_7 &= 2q \quad (Q_d). \end{aligned}$$

Moreover, the lines $l_a, \ldots, l_d$ give the following relations

$$\begin{aligned} p_3 + p_4 + p_7 &= q \quad (l_a), \\ p_3 + p_5 + p_8 &= q \quad (l_b), \\ p_5 + p_6 + p_7 &= q \quad (l_c), \\ p_4 + p_6 + p_8 &= q \quad (l_d). \end{aligned}$$

Now, we can gather all the information we have and conclude. First we will show that $x_b = x_a$: we start by subtracting (l_a) from (Q_b). This gives the following:

$$x_b + p_1 + p_2 + p_6 - p_3 = q.$$

We can now add (l_b) and obtain

$$x_b + p_1 + p_2 + p_5 + p_6 + p_8 = 2q,$$

which implies $x_a = x_b$, after subtracting (Q_a). In a similar way we can conclude that $x_a = x_c$ (computing $(Q_c) - (l_a) + (l_c) - (Q_a)$), and $x_a = x_d$ (computing $(Q_d) - (l_a) + (l_d) - (Q_a)$). Hence $x_a = x_b = x_c = x_d$ and concludes the proof.

C.5.1. Torsion. This the best place to prove that the construction made in section 2.7.1 actually leads to a rational elliptic surface with torsion. We will use the notation introduced in subsection C.1.3. We will show that for every elliptic curve (E, O) passing through $p_0, \ldots, p_8$ there exists an explicit point $p \neq O$ such that $2 \cdot p = O$.

It is enough to use the relations and the fact that $p_0 = x_a = x_b = x_c = x_d$, exhibited previously in this section. We first take (Q_b), subtract (l_d) and (l_c) and get

$$\begin{aligned} O &= p_0 + p_1 + p_2 + p_4 + p_6 + p_7 - (p_4 + p_6 + p_8) - (p_5 + p_6 + p_7) \\ &= p_0 + p_1 + p_2 - p_5 - p_6 - p_8. \end{aligned}$$

Adding (Q_a) we get

$$2 \cdot (p_0 + p_1 + p_2) = 2q;$$

This is equivalent to

$$2 \cdot (p_0 + p_1 + p_2 - q) = O,$$

so the point $p = p_0 + p_1 + p_2 - q$ is a 2-torsion point for every (E, O) in the pencil determined by $p_0, \ldots, p_8$.

We still have to show that p is not O itself. Since p_0, p_1 and p_2 are not collinear, we have that $p_0 + p_1 + p_2 \neq q$. This implies that $p \neq O$.

Similarly, one can prove that $2 \cdot (p_3 - p_6) = O$, $2 \cdot (p_4 - p_5) = O$ and $2 \cdot (p_7 - p_8) = O$.

APPENDIX D

Four Reducible Fibers: the Torsion Case

D.1. Concurrency

From the construction of a rational elliptic surface with four reducible fibers, given by four reducible members in a linear pencil of cubics, it remains to prove that $l_c \cap \beta = \alpha \cap \beta$ and $l_d \cap \beta = \delta \cap \beta$. We will prove it now, using some properties of the projective plane. We use the same notation used in that construction.

Up to projectivities, we can always choose

$$p_1 = (0,1,0), \quad p_2 = (0,0,1), \quad c = (1,0,0).$$

Then, the points a and b are of the following forms:

$$a = (1, a_x, a_y), \quad b = (1, b_x, b_y).$$

We can exclude that a and b have zero as the first coordinate, since they are general points on the plane (so not collinear to p_1 and p_2). In this setting, the lines α, β, δ are given by Plücker equations:

$$\alpha = (-a_y b_x, a_y, -a_x + b_x), \quad \beta = (a_x b_y, -b_y, -a_x), \quad \delta = (a_y b_x, -a_y + b_y, -b_x).$$

These allow us to compute the points $\alpha \cap \beta$ and $\beta \cap \delta$:

$$\begin{aligned} \alpha \cap \beta &= (-a_x a_y - a_x b_y + b_x b_y, -a_x a_y b_x - a_x^2 b_y + a_x b_x b_y, -a_x a_y b_y + a_y b_x b_y), \\ \beta \cap \delta &= (-a_x a_y + a_x b_y + b_x b_y, -a_x a_y b_x + a_x b_x b_y, -a_x a_y b_y + a_y b_x b_y + a_x b_y^2). \end{aligned}$$

Now, we can deduce the equations for $p_0 \cup p_1$ and $p_0 \cup p_2$:

$$\begin{aligned} p_0 \cup p_1 &= (-a_x a_y b_y + a_y b_x b_y + a_x b_y^2, 0, a_x a_y - a_x b_y - b_x b_y), \\ p_0 \cup p_2 &= (a_x a_y b_x + a_x^2 b_y - a_x b_x b_y, -a_x a_y - a_x b_y + b_x b_y, 0). \end{aligned}$$

Intersecting $p_0 \cup p_1$ with α and $p_0 \cup p_2$ with δ, we get two points that allow us to construct l_a and l_b, given by:

$$\begin{aligned} l_a = (&a_x a_y^3 b_x + a_x^2 a_y^2 b_y - 4 a_x a_y^2 b_x b_y + a_y^2 b_x^2 b_y - a_x^2 a_y b_y^2 + 2 a_x a_y b_x b_y^2, \\ &- a_x a_y^3 + 2 a_x a_y^2 b_y - a_x a_y b_y^2, -a_x^2 a_y b_y + 2 a_x a_y b_x b_y - a_y b_x^2 b_y + a_x^2 b_y^2 - a_x b_x b_y^2), \\ l_b = (&-a_x a_y^2 b_x^2 - 2 a_x^2 a_y b_x b_y + 4 a_x a_y b_x^2 b_y - a_y b_x^3 b_y + a_x^2 b_x b_y^2 - a_x b_x^2 b_y^2, \\ &a_x a_y^2 b_x + a_x^2 a_y b_y - 2 a_x a_y b_x b_y - a_x^2 b_y^2 + a_x b_x b_y^2, a_x^2 b_x b_y - 2 a_x b_x^2 b_y + b_x^3 b_y). \end{aligned}$$

Now we can compute the lines l_c and l_d:

$$\begin{aligned} l_c = (&-a_x a_y^3 b_x^2 - 2 a_x^2 a_y^2 b_x b_y + 4 a_x a_y^2 b_x^2 b_y - a_y^2 b_x^3 b_y + a_x^2 a_y b_x b_y^2 - a_x a_y b_x^2 b_y^2, \\ &a_x a_y^3 b_x + a_x^2 a_y^2 b_y - 2 a_x a_y^2 b_x b_y - a_x^2 a_y b_y^2 + a_x a_y b_x b_y^2, \\ &- a_x^3 a_y b_y + 4 a_x^2 a_y b_x b_y - 4 a_x a_y b_x^2 b_y + a_y b_x^3 b_y + a_x^3 b_y^2 - 2 a_x^2 b_x b_y^2 + a_x b_x^2 b_y^2), \\ l_d = (&-a_x a_y^3 b_x^2 - a_x^2 a_y^2 b_x b_y + 4 a_x a_y^2 b_x^2 b_y - a_y^2 b_x^3 b_y + a_x^2 a_y b_x b_y^2 - 2 a_x a_y b_x^2 b_y^2, \\ &a_x a_y^3 b_x + a_x^2 a_y^2 b_y - 4 a_x a_y^2 b_x b_y - 2 a_x^2 a_y b_y^2 + 4 a_x a_y b_x b_y^2 + a_x^2 b_y^3 - a_x b_x b_y^3, \\ &a_x^2 a_y b_x b_y - 2 a_x a_y b_x^2 b_y + a_y b_x^3 b_y - a_x^2 b_x b_y^2 + a_x b_x^2 b_y^2). \end{aligned}$$

Now we can check if $l_c \cap \beta = \alpha \cap \beta$ and $l_d \cap \beta = \delta \cap \beta$: this is true, since

$$l_c \cap \beta = (-a_x a_y - a_x b_y + b_x b_y, -a_x a_y b_x - a_x^2 b_y + a_x b_x b_y, -a_x a_y b_y + a_y b_x b_y),$$
$$l_d \cap \beta = (-a_x a_y + a_x b_y + b_x b_y, -a_x a_y b_x + a_x b_x b_y, -a_x a_y b_y + a_y b_x b_y + a_x b_y^2).$$

This concludes the proof.

D.2. Members

In this section we will show that the construction in subsection 2.7.1 leads to a linear pencil of cubics with four reducible members, all given by a line and an irreducible conic.

We will use Pascal's Theorem several times: consider p_0, p_1, p_2 and the three intersection points given by 3 of the four green lines. Those six points determine an hexagon (meaning a 6-tuple of points); from the configuration of the base points, the opposite sides of this hexagon meet at three points on one of the blue lines; this implies that the hexagon is inscribed into a conic. In the following drawings we underline each hexagon in red and label its points in order, so that the intersection points between opposite sides are given by

$$(n \cup n+1) \cap (n+3 \cup n+4), \text{ all modulo } 6.$$

The first hexagon is given by $(p_2, p_0, p_1, p_5, p_6, p_8)$ and the opposite sides meet at $\beta \cap l_c$, $\beta \cap l_d$ and $(p_1 \cup c) \cap (p_2 \cup a)$. These three points are collinear, since they belong to β.

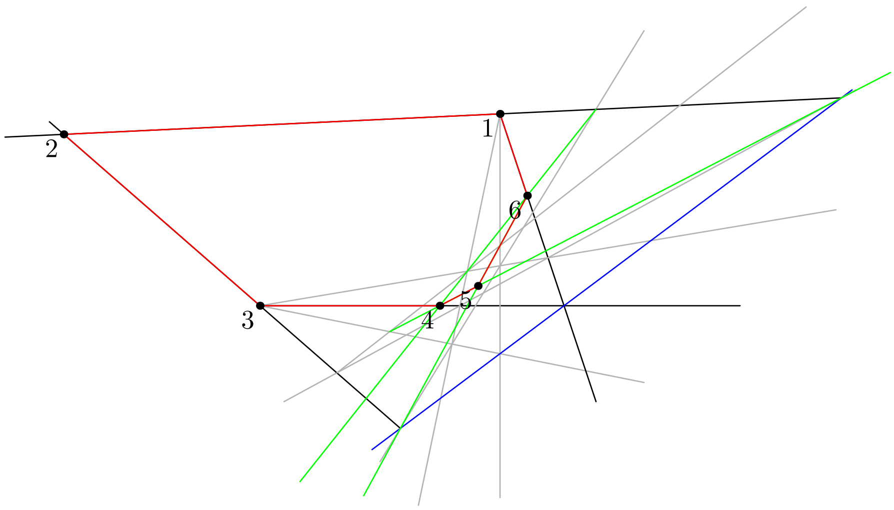


Figure D.1. A reducible member.

We will denote by Q_a the conic passing through $p_0, p_1, p_2, p_5, p_6, p_8$.

The second hexagon is given by $(p_2, p_0, p_1, p_7, p_6, p_4)$ and the opposite sides meet at $\beta \cap l_c$, $\beta \cap l_d$ and $(p_2 \cup c) \cap (p_1 \cup b)$. These three points are collinear, since they belong to β.

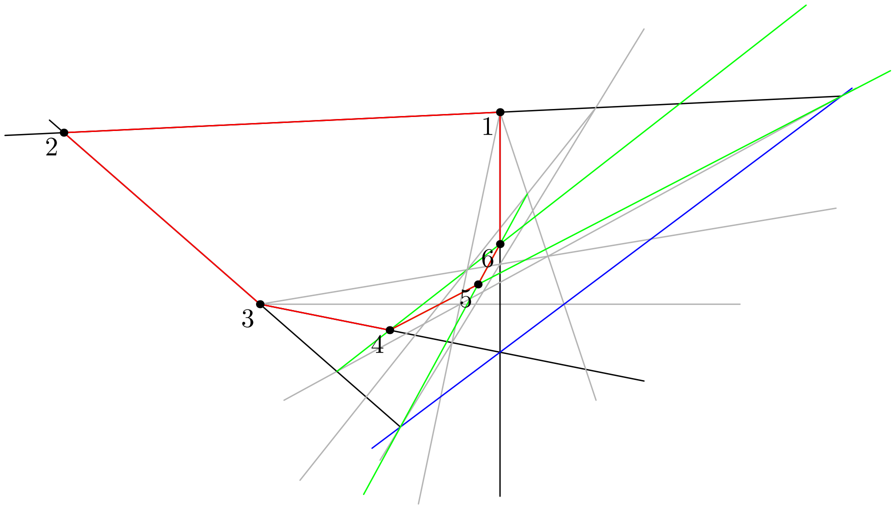


FIGURE D.2. A reducible member.

We will denote by Q_b the conic passing through $p_0, p_1, p_2, p_4, p_6, p_7$.

The third hexagon is given by $(p_2, p_0, p_1, p_3, p_8, p_4)$ and the opposite sides meet at $\delta \cap l_b$, $\delta \cap l_d$ and $(p_1 \cup a) \cap (p_2 \cup c)$. These three points are collinear, since they belong to δ.

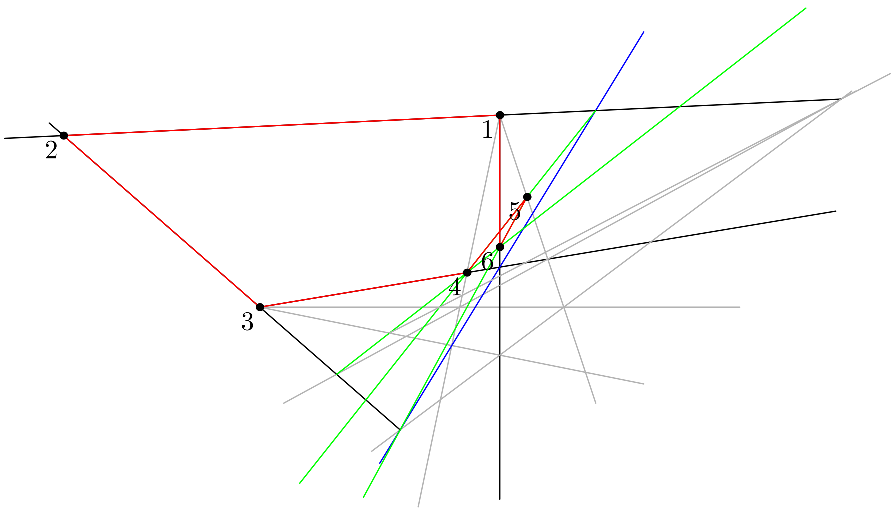


FIGURE D.3. A reducible member.

We will denote by Q_c the conic passing through $p_0, p_1, p_2, p_3, p_4, p_8$.

The fourth hexagon is given by $(p_2, p_0, p_1, p_5, p_3, p_7)$ and the opposite sides meet at $\alpha \cap l_c$, $\alpha \cap l_a$ and $(p_1 \cup c) \cap (p_2 \cup b)$. These three points are collinear, since they belong to α.

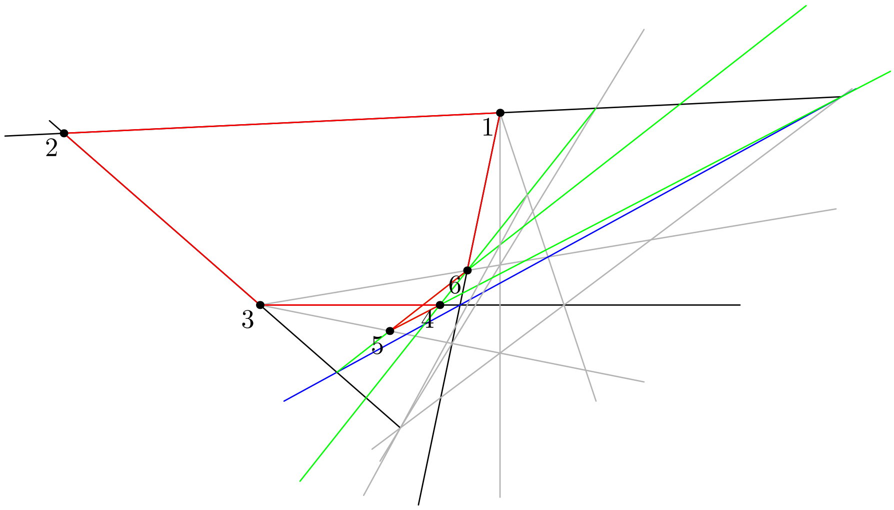


FIGURE D.4. A reducible member.

We will denote by Q_d the conic passing through $p_0, p_1, p_2, p_3, p_5, p_7$.

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