GRAPH THEORY – LECTURE 1
INTRODUCTION TO GRAPH MODELS

Abstract. Chapter 1 introduces some basic terminology. §1.1 is concerned with the existence and construction of a graph with a given degree sequence. §1.2 presents some families of graphs to which frequent reference occurs throughout the course. §1.4 introduces the notion of distance, which is fundamental to many applications. §1.5 introduces paths, trees, and cycles, which are critical concepts to much of the theory.

Outline

1. Graphs and Digraphs

terminology for graphical objects

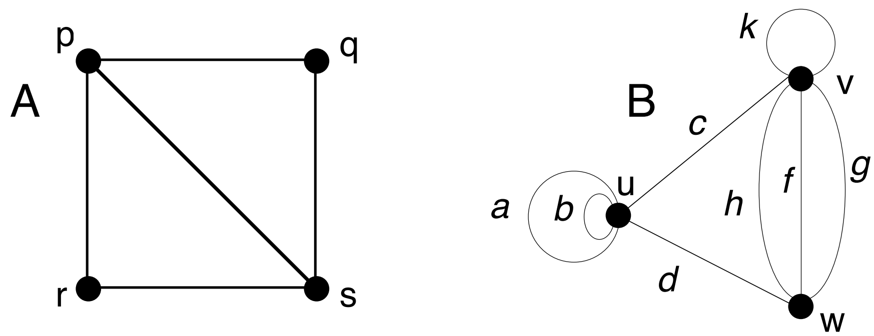

Figure 1.1: ***Simple graph*** A; ***graph*** B.

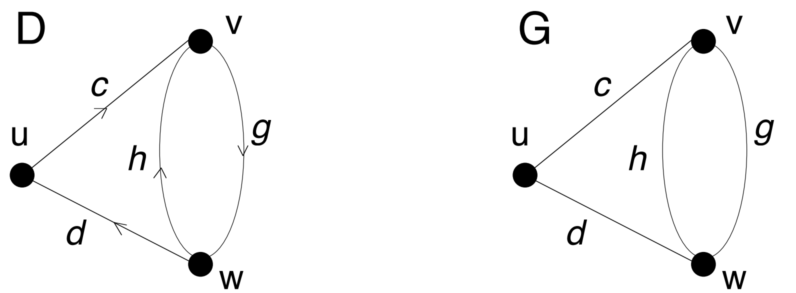

Figure 1.3: ***Digraph*** D; its ***underlying graph*** G.

Degree

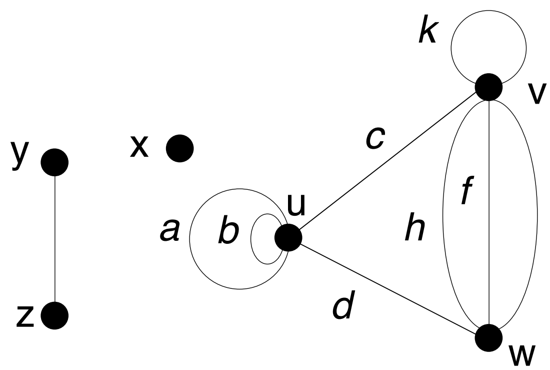

Figure 1.9: A graph with ***degree sequence*** 6, 6, 4, 1, 1, 0.

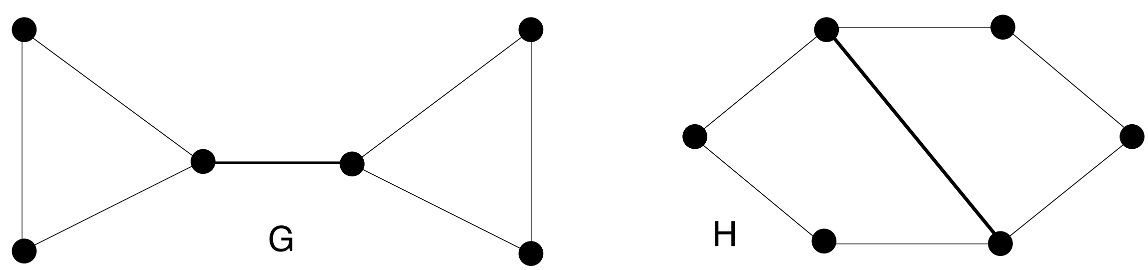

Figure 1.10: Both degree sequences are $\langle 3, 3, 2, 2, 2, 2 \rangle$.

Proposition 1.1. *A non-trivial simple graph G must have at least one pair of vertices whose degrees are equal.*

Proof. pigeonhole principle □

Theorem 1.2 (***Euler's Degree-Sum Thm***)**.** *The sum of the degrees of the vertices of a graph is twice the number of edges.*

Corollary 1.3. *In a graph, the number of vertices having odd degree is an even number.*

Corollary 1.4. *The degree sequence of a graph is a finite, non-increasing sequence of nonnegative integers whose sum is even.*

General Graph with Given Degree Sequence

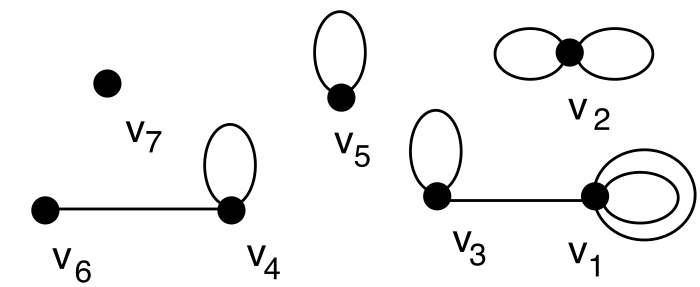

Figure 1.11: General graph with deg seq $\langle 5, 4, 3, 3, 2, 1, 0 \rangle$.

Simple Graph with Given Degree Sequence

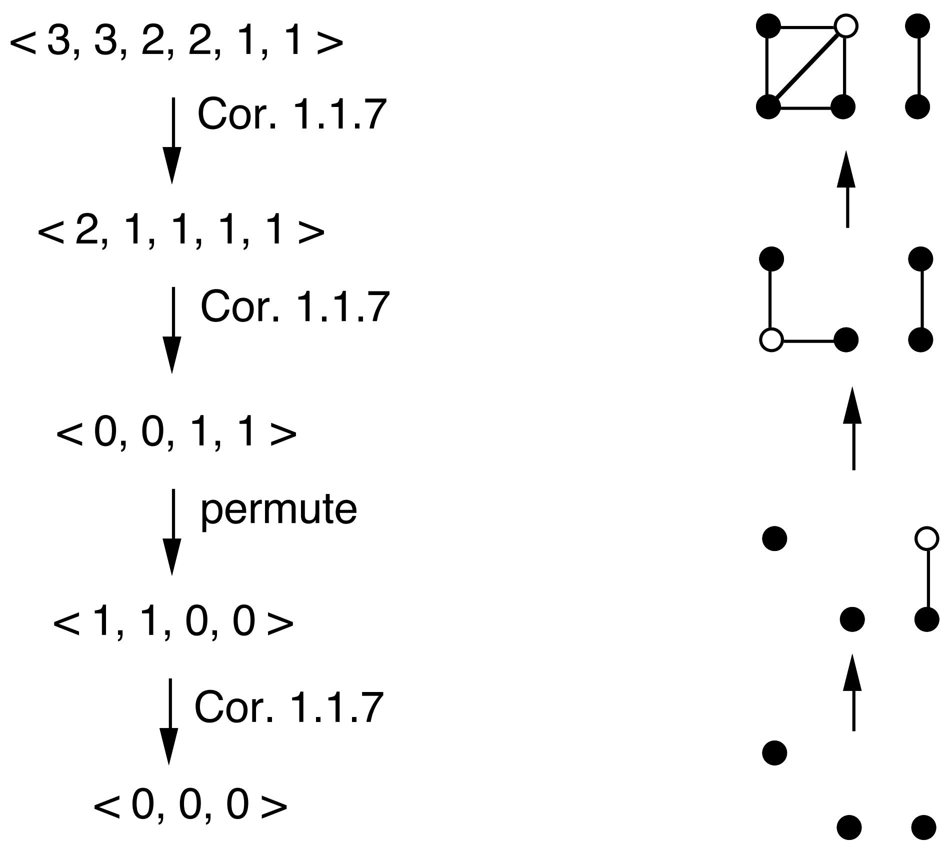

Figure 1.13: Simple graph with deg seq $\langle 3, 3, 2, 2, 1, 1 \rangle$.

Havel-Hakimi Theorem

Theorem 1.6. *Let* $\langle d_1, d_2, \ldots, d_n \rangle$ *be a graphic sequence, with*

$$d_1 \geq d_2 \geq \ldots \geq d_n$$

Then there is a simple graph with vertex-set $\{v_1, \ldots, v_n\}$ *s.t.*

$$deg(v_i) \; = \; d_i \quad \text{for } i = 1, 2, \ldots, n$$

with v_1 *adjacent to vertices* $v_2, \ldots, v_{d_1+1}$.

Proof. Among all simple graphs with vertex-set

$$V = \{v_1, v_2, \ldots, v_n\} \text{ and } deg(v_i) = d_i : i = 1, 2, \ldots, n$$

let G be a graph for which the number

$$r \; = \; |N_G(v_1) \cap \{v_2, \ldots, v_{d_1+1}\}|$$

is maximum. If $r = d_1$, then the conclusion follows.

Alternatively, if $r < d_1$, then there is a vertex

$$v_s : \quad 2 \; \leq \; s \; \leq \; d_1 + 1$$

such that v_1 is not adjacent to v_s, and $\exists$ vertex

$$v_t : \quad t \; > \; d_1 + 1$$

such that v_1 is adjacent to v_t (since $deg(v_1) = d_1$).

Moreover, since $deg(v_s) \geq deg(v_t)$, $\exists$ vertex v_k such that v_k is adj to v_s but not to v_t, as on the left of Fig 1.14. Let $\widetilde{G}$ be the graph obtained from G by replacing edges v_1v_t and v_sv_k with edges v_1v_s and v_tv_k, as on the right of Fig 1.14, so all degrees are all preserved.

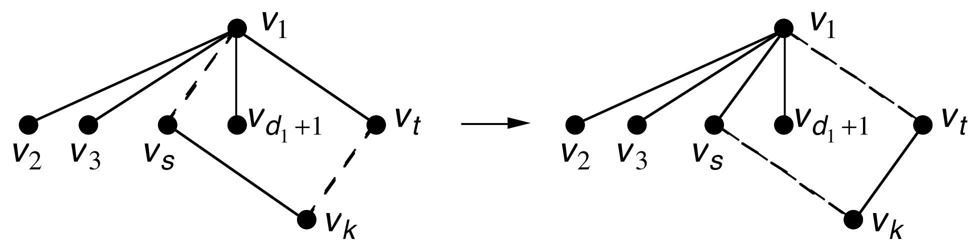

Figure 1.14: Switching adjacencies while preserving all degrees.

Thus, $|N_{\widetilde{G}}(v_1) \cap \{v_2, \ldots, v_{d_1+1}\}| = r + 1$, which contradicts the choice of graph G. $\square$

Corollary 1.7 (Havel (1955) and Hakimi (1961))**.** *A sequence $\langle d_1, d_2, \ldots, d_n \rangle$ of nonneg ints, such that $d_1 \geq d_2 \geq \ldots \geq d_n$, is graphic if and only if the sequence*

$$\left\langle d_2 - 1, \ \ldots, \ d_{d_1+1} - 1, \ d_{d_1+2}, \ \ldots, \ d_n \right\rangle$$

is graphic. (See Exercises for proof.)

Remark 1.1. Cor 1.7 yields a recursive algorithm that decides whether a non-increasing sequence is graphic.

ALGORITHM: RECURSIVE GRAPHICSEQUENCE($\langle d_1, d_2, \ldots, d_n \rangle$)
Input: a non-increasing sequence $\langle d_1, d_2, \ldots, d_n \rangle$.
Output: TRUE if the sequence is graphic; FALSE if it is not.

```
If d_1 = 0
    Return TRUE
Else
    If d_n < 0
        Return FALSE
    Else
        Let ⟨a_1, a_2, ..., a_{n-1}⟩ be a non-incr permutation
          of ⟨d_2 - 1, ..., d_{d_1+1} - 1, d_{d_1+2}, ..., d_n⟩.
        Return GraphicSequence(⟨a_1, a_2, ..., a_{n-1}⟩)
```

2. Families of Graphs

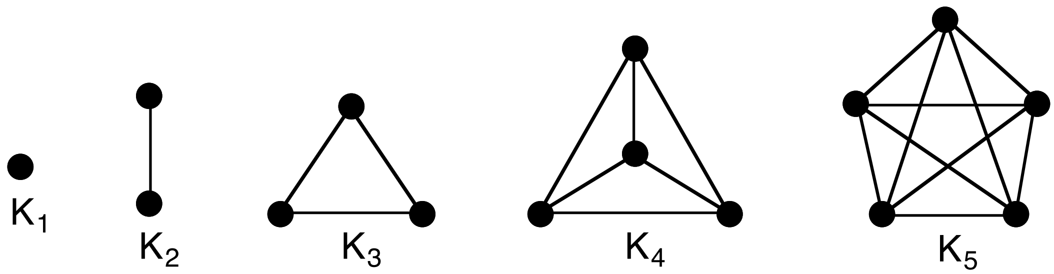

Figure 2.1: The first five ***complete graphs***.

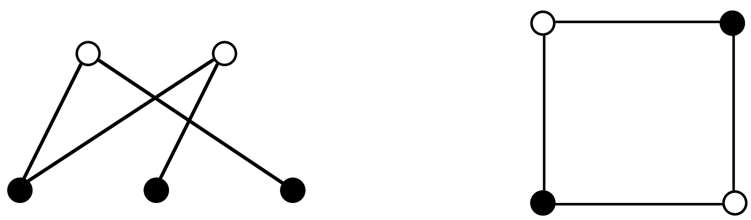

Figure 2.2: Two ***bipartite graphs***.

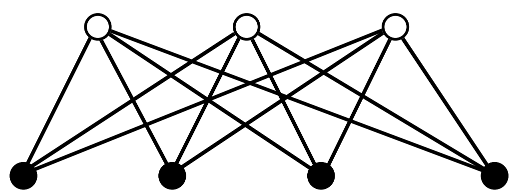

Figure 2.4: The ***complete bipartite graph*** $K_{3,4}$.

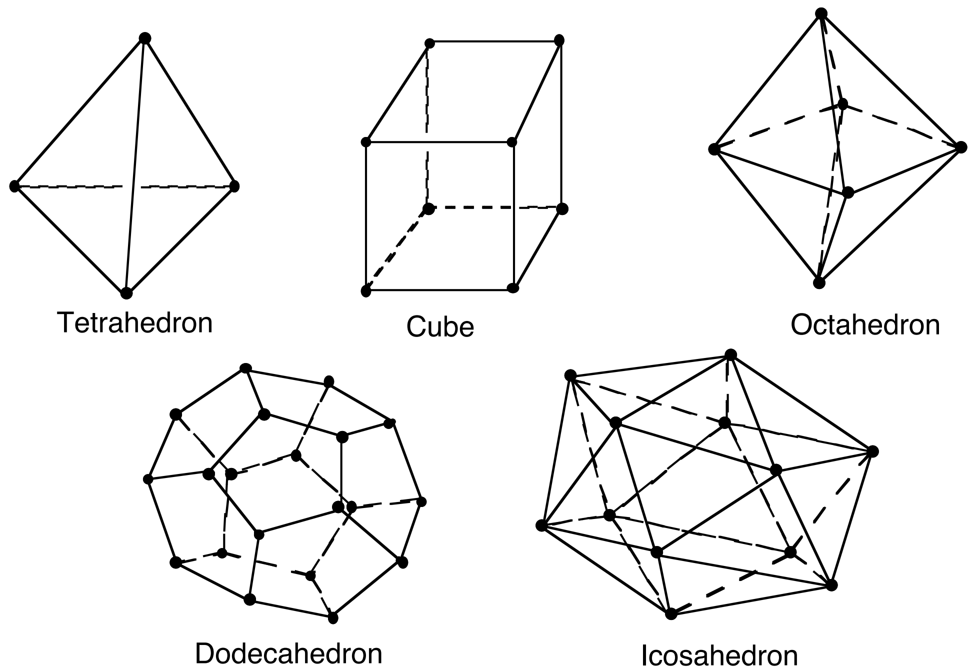

Figure 2.5: The five ***platonic graphs***.

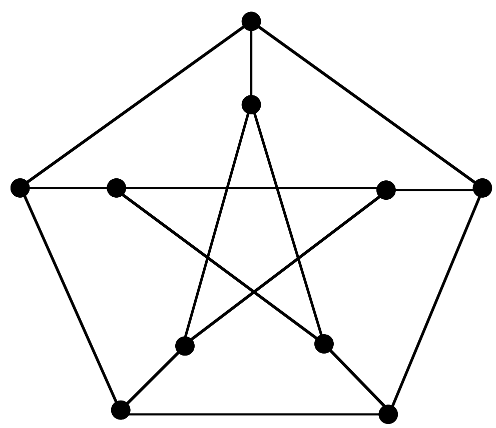

Figure 2.6: The ***Petersen graph***.

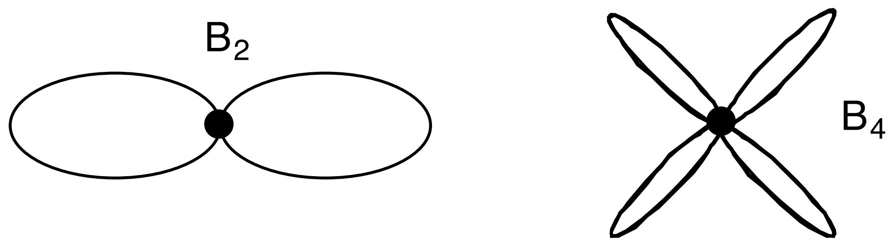

Figure 2.8: ***Bouquets*** B_2 and B_4.

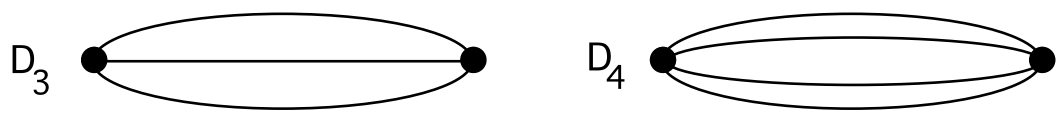

Figure 2.9: The ***Dipoles*** D_3 and D_4.

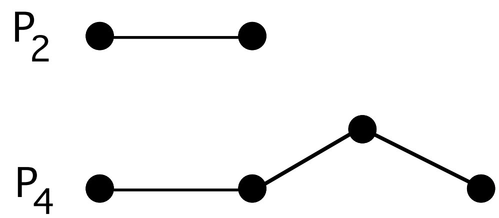

Figure 2.10: ***Path graphs*** P_2 and P_4.

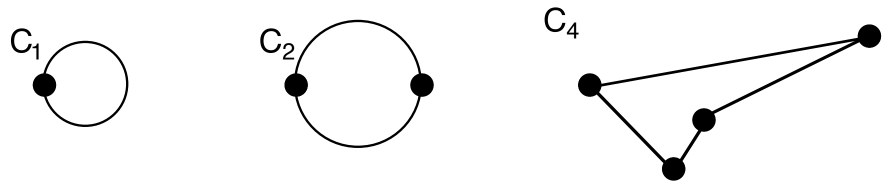

Figure 2.11: ***Cycle graphs*** C_1, C_2, and C_4.

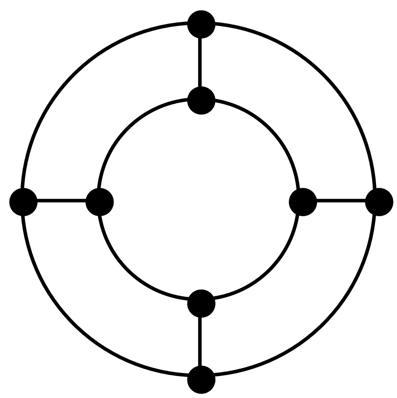

Figure 2.12: ***Circular ladder graph*** CL_4.

Circulant Graphs

Def 2.1. To the group of integers

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$$

under addition modulo n and a set

$$S \subseteq \{1, \ldots, n-1\}$$

we associate the ***circulant graph***

$$circ(n : S)$$

whose vertex set is $\mathbb{Z}_n$, such that two vertices i and j are adjacent if and only if there is a number $s \in S$ such that $i + s = j \mod n$ or $j + s = i \mod n$. In this regard, the elements of the set S are called ***connections***.

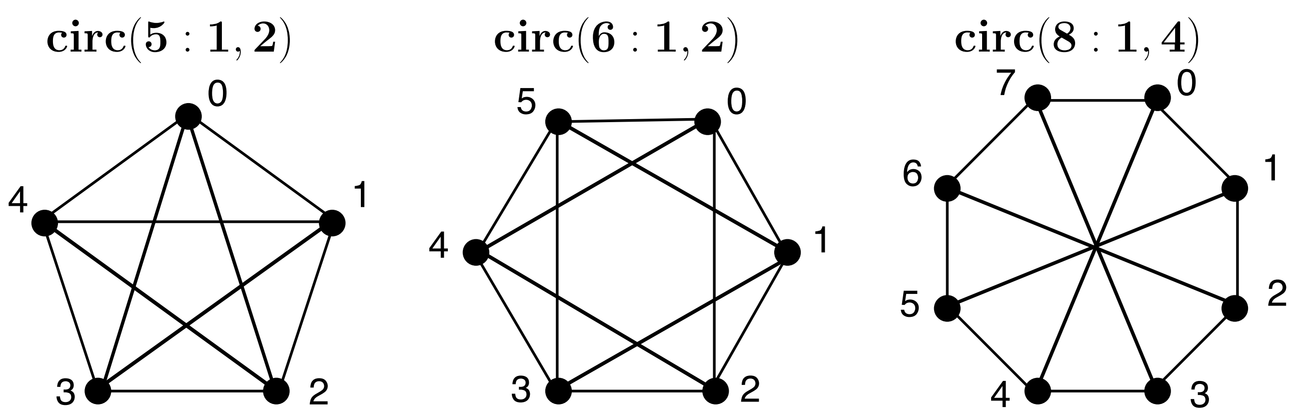

Figure 2.13: Three circulant graphs.

Intersection and Interval Graphs

Def 2.2. A simple graph G with vertex set

$$V_G = \{v_1, v_2, \ldots, v_n\}$$

is an ***intersection graph*** if there exists a family of sets

$$\mathcal{F} \; = \; \{S_1, S_2, \ldots, S_n\}$$

s. t. vertex v_i is adjacent to v_j if and only $i \neq j$ and $S_i \cap S_j \neq \emptyset$.

Def 2.3. A simple graph G is an ***interval graph*** if it is an intersection graph corresponding to a family of intervals on the real line.

Example 2.1. The graph G in Figure 2.14 is an interval graph for the following family of intervals:

$$a \leftrightarrow (1,3) \qquad b \leftrightarrow (2,6) \qquad c \leftrightarrow (5,8) \qquad d \leftrightarrow (4,7)$$

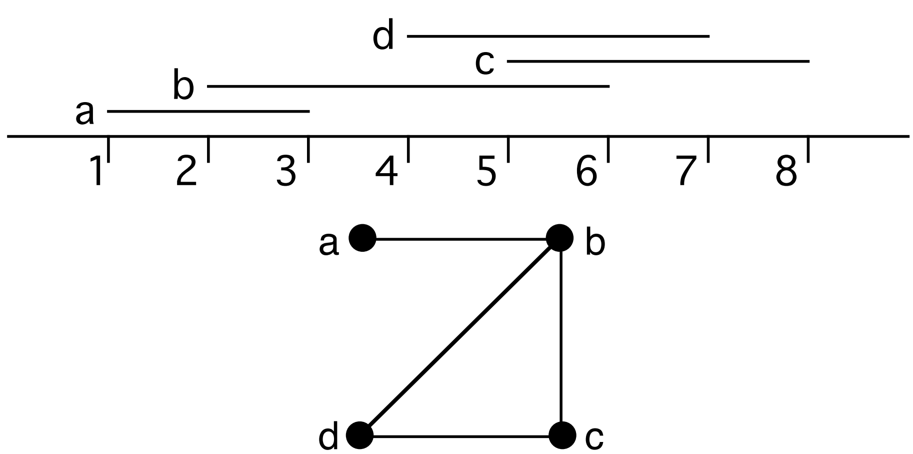

Figure 2.14: An interval graph.

Line Graphs

Line graphs are a special case of intersection graphs.

Def 2.4. The ***line graph*** $L(G)$ of a graph G has a vertex for each edge of G, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have a vertex in common.

Thus, the line graph $L(G)$ is the intersection graph corresponding to the endpoint sets of the edges of G.

Example 2.2. Figure 2.15 shows a graph G and its line graph $L(G)$.

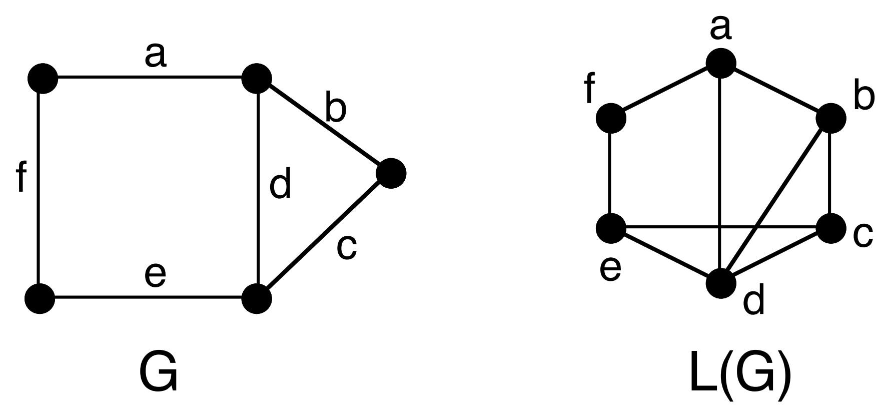

Figure 2.15: A graph and its line graph.

4. Walks and Distance

Def 4.1. A ***walk*** from v_0 to v_n is an alternating sequence

$$W = \langle v_0, \ e_1, \ v_1, \ e_2, \ ..., \ v_{n-1}, \ e_n, \ v_n \rangle$$

of vertices and edges, such that

$$endpts(e_i) = \{v_{i-1}, \ v_i\}, \quad \text{for } i = 1, ..., n$$

In a simple graph, there is only one edge beween two consecutive vertices of a walk, so one could abbreviate the walk as

$$W = \langle v_0, \ v_1, \ \ldots, \ v_n \rangle$$

In a general graph, one might abbreviate as

$$W = \langle v_0, \ e_1, \ e_2, \ ..., \ e_n, \ v_n \rangle$$

Def 4.2. The ***length*** of a walk or directed walk is the number of edge-steps in the walk sequence.

Def 4.3. A walk of length zero, i.e., with one vertex and no edges, is called a ***trivial walk***.

Def 4.4. A ***closed walk*** (or ***closed directed walk***) is a nontrivial walk (or directed walk) that begins and ends at the same vertex. An ***open walk*** (or ***open directed walk***) begins and ends at different vertices.

Def 4.5. The ***distance*** $d(s, t)$ from a vertex s to a vertex t in a graph G is the length of a shortest s-t walk if one exists; otherwise, $d(s, t) = \infty$.

Eccentricity, Diameter, and Radius

Def 4.6. The ***eccentricity*** of a vertex v, denoted $ecc(v)$, is the distance from v to a vertex farthest from v. That is,

$$ecc(v) = \max_{x \in V_G}\{d(v,x)\}$$

Def 4.7. The ***diameter*** of a graph is the max of its eccentricities, or, equivalently, the max distance between two vertices. i.e.,

$$diam(G) = \max_{x \in V_G}\{ecc(x)\} = \max_{x,y \in V_G}\{d(x,y)\}$$

Def 4.8. The ***radius*** of a graph G, denoted $rad(G)$, is the min of the vertex eccentricities. That is,

$$rad(G) = \min_{x \in V_G}\{ecc(x)\}$$

Def 4.9. A ***central vertex*** v of a graph G is a vertex with min eccentricity. Thus, $ecc(v) = rad(G)$.

Example 4.7. The graph of Fig 4.7 below has diameter 4, achieved by the vertex pairs u, v and u, w. Vertices x and y have eccentricity 2 and all other vertices have greater eccentricity. Thus, the graph has radius 2 and central vertices x and y.

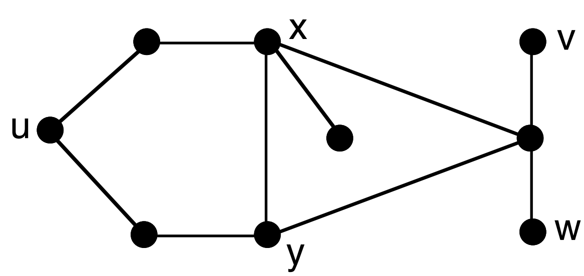

Figure 4.7: A graph with diameter 4 and radius 2.

Connectedness

Def 4.10. Vertex v is ***reachable from*** vertex u if there is a walk from u to v.

Def 4.11. A graph is ***connected*** if for every pair of vertices u and v, there is a walk from u to v.

Def 4.12. A digraph is ***connected*** if its underlying graph is connected.

Example 4.8. The non-connected graph in Figure 4.8 is made up of connected pieces called *components*. See §2.3.

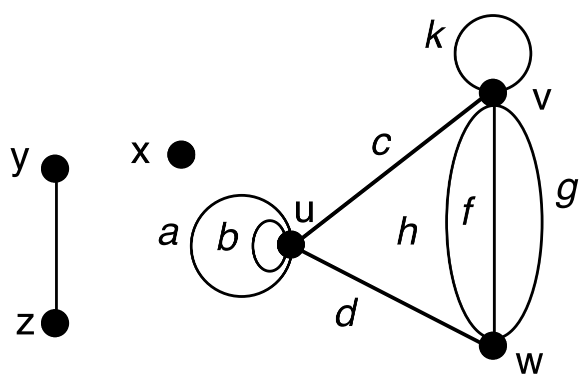

Figure 4.8: Non-connected graph with three components.

5. Paths, Cycles, and Trees

Def 5.1. A ***trail*** is a walk with no repeated edges.

Def 5.2. A ***path*** is a trail with no repeated vertices (except possibly the initial and final vertices).

Def 5.3. A walk, trail, or path is ***trivial*** if it has only one vertex and no edges.

Example 5.1. In Fig 5.1, $W = \langle v, a, e, f, a, d, z \rangle$ is the edge sequence of a walk but not a trail, because edge a is repeated, and $T = \langle v, a, b, c, d, e, u \rangle$ is a trail but not a path, because vertex x is repeated.

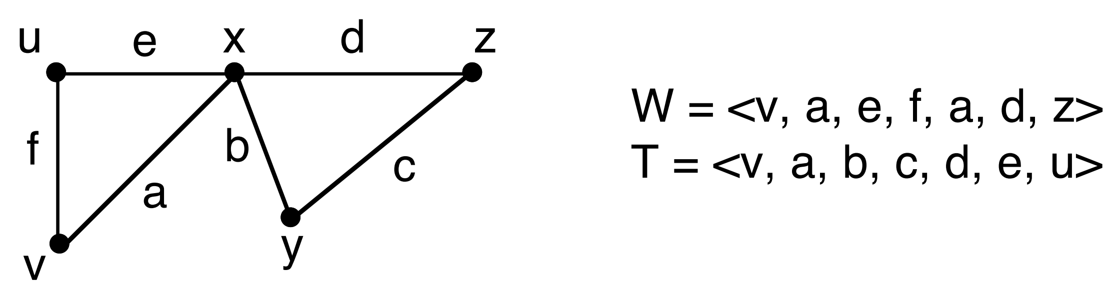

Figure 5.1: Walk W is not a trail; trail T is not a path.

Cycles

Def 5.4. A nontrivial closed path is called a ***cycle***. It is called an ***odd cycle*** or an ***even cycle***, depending on the parity of its length.

Def 5.5. An ***acyclic graph*** is a graph that has no cycles.

Eulerian Graphs

Def 5.6. An ***eulerian trail*** in a graph is a trail that contains every edge of that graph.

Def 5.7. An ***eulerian tour*** is a closed eulerian trail.

Def 5.8. An ***eulerian graph*** is a graph that has an eulerian tour.

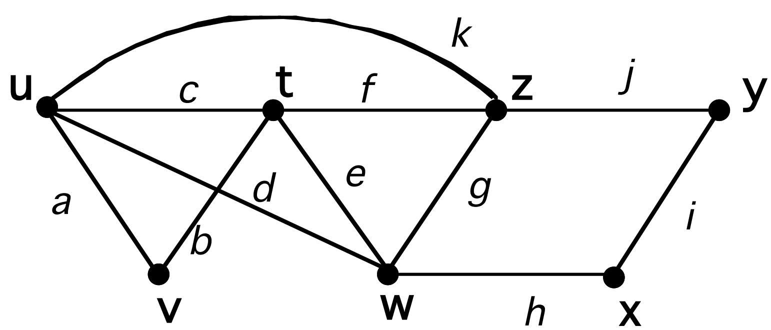

Figure 5.6: An eulerian graph.

Hamiltonian Graphs

Def 5.9. A cycle that includes every vertex of a graph is call a ***hamiltonian cycle***.

Def 5.10. A ***hamiltonian graph*** is a graph that has a hamiltonian cycle. (§6.3 elaborates on hamiltonian graphs).

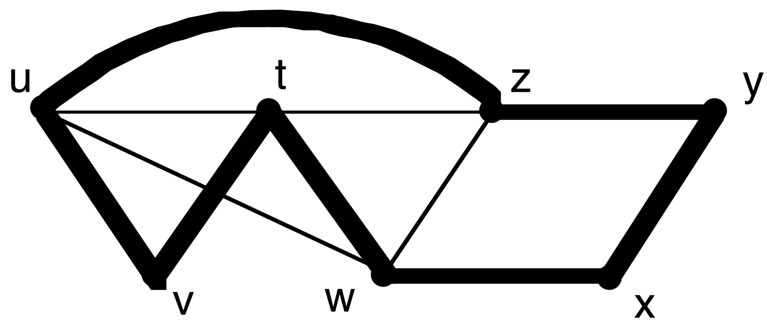

Figure 5.3: An hamiltonian graph.

Girth

Def 5.11. The ***girth*** of a graph with at least one cycle is the length of a shortest cycle. The girth of an acyclic graph is undefined.

Example 5.2. The girth of the graph in Figure 5.7 is 3 since there is a 3-cycle but no 2-cycle or 1-cycle.

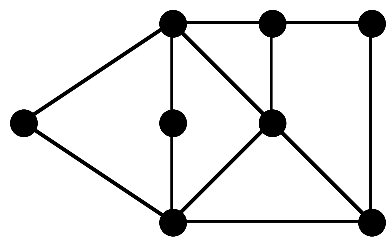

Figure 5.7: A graph with girth 3.

Trees

Def 5.12. A ***tree*** is a connected graph that has no cycles.

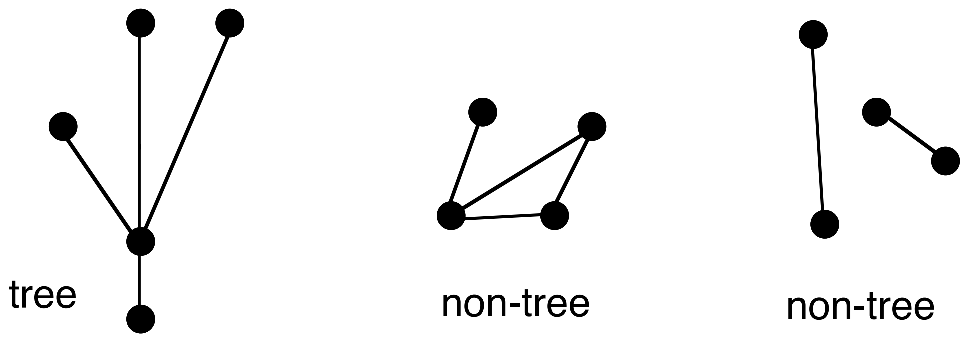

Figure 5.8: A tree and two non-trees.

Theorem 5.4. *A graph G is bipartite iff it has no odd cycles.*

Proof. Nec ($\Rightarrow$): Suppose G is bipartite. Since traversing each edge in a walk switches sides of the bipartition, it requires an even number of steps for a walk to return to the side from which it started. Thus, a cycle must have even length.

Suff ($\Leftarrow$): Let G be a graph with $n \geq 2$ vertices and no odd cycles. W.l.o.g., assume that G is connected. Pick any vertex u of G, and define a partition (X, Y) of V as follows:

$$X = \{x \mid d(u, x) \text{ is even}\};\ Y = \{y \mid d(u, y) \text{ is odd}\}$$

Suppose two vertices v and w in one of the sets are joined by an edge e. Let P_1 be a shortest u-v path, and let P_2 be a shortest u-w path. By definition of the sets X and Y, the lengths of these paths are both even or both odd. Starting from vertex u, let x be the last vertex common to both paths (see Fig 5.9).

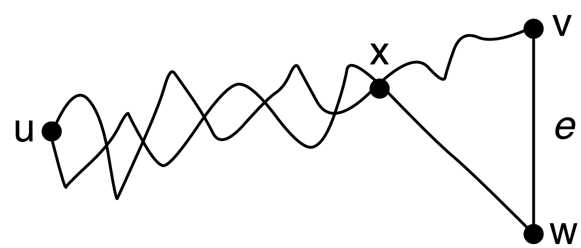

Figure 5.9: Figure for suff part of Thm 5.4 proof.

Since P_1 and P_2 are both shortest paths, their $u \to x$ sections have equal length. Thus, the lengths of the $x \to v$ section of P_1 and the $x \to w$ section of P_2 are either both even or both odd. But then the concatenation of those two sections with the edge e forms an odd cycle, contradicting the hypothesis. Hence, (X, Y) is a bipartition of G. $\square$

7. Supplementary Exercises

Exercise 1 A 20-vertex graph has 62 edges. Every vertex has degree 3 or 7. How many vertices have degree 3?

Exercise 8 How many edges are in the hypercube graph Q_4?

Exercise 11 In the circulant graph $circ(24 : 1, 5)$, what vertices are at distance 2 from vertex 3?

Def 7.1. The ***edge-complement*** of a simple graph G is the simple graph $\overline{G}$ on the same vertex set such that two vertices of $\overline{G}$ are adjacent if and only if they are *not* adjacent in G.

Exercise 20 Let G be a simple bipartite graph with at least 5 vertices. Prove that $\overline{G}$ is not bipartite. (See §2.4.)