CS Theory (Spring '24)	Assigned: January 22, 2024
Homework 1	
Instructor: *Josh Alman*	Due: January 31, 2024 at 11:59 pm

Collaboration on homework is allowed. However, you must write up solutions by yourself and understand everything that you hand in. You may also consult other reference materials, but you may not seek out answers from other sources.

Write the solution to each problem on a separate page. Please do not write your name on any page of the submission; we are using anonymous grading in GradeScope.

Be sure to answer the final question listing your collaborators and other sources used. See the course webpage for more details.

There are 120 total possible points, and also bonus questions worth an additional 20 points.

1 A Fun Card Game (20 points)

Alice and Bob play the following game: First Bob picks a positive integer n, then Alice picks a positive integer k (that can depend on n). Bob then places n^2 cards face-down in an n by n grid. Each card has a 0 or a 1 on it, which Bob gets to choose, and these are hidden to Alice since the cards are face-down. Alice then gets to choose any k of the cards to reveal, so she sees just the 0 or 1 written on those cards and not the others. She looks at those k cards, then chooses any length-n string w over the alphabet $\{0,1\}$ (in other words, a $w \in \{0,1\}^n$) and tells it to Bob. Finally, Bob reveals the rest of the cards. If the cards in sequence in any row of Bob's grid are equal to w, then Bob wins. Otherwise, Alice wins.

What is the smallest k that Alice can pick (that can depend on n) so that she can always win the game? Prove the correctness of your answer. (Note that this means you need to give a strategy and prove it is correct for that choice of k, and you also have to prove that there is no strategy that works for any smaller choice of k.)

2 Constructing DFAs (30 points; each part 10 points)

For each of the following languages over the alphabet $\Sigma = \{0,1\}$, give a DFA that recognizes the language, and explain why it's correct. Try to use as few states as possible.

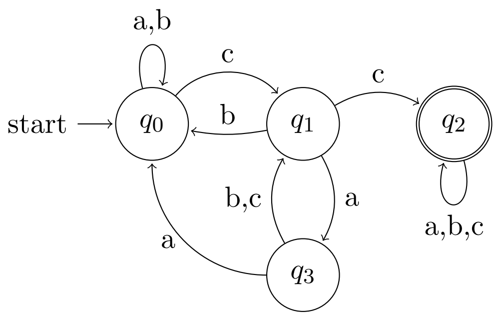
(a) $\{w \in \Sigma^* \mid w \text{ has an odd number of 1s}\}$. For instance, 11111 has five 1s and is in the language, and 001000 has one 1 and is in the language, while 11001010 has four 1s and is not in the language.

(b) $\{w \in \Sigma^* \mid w \text{ starts and ends with the same character}\}$. For instance, 100101 starts and ends with 1 and is in the language, and 001000 starts and ends with 0 and is in the language, while 11001010 starts with 1 but ends with 0 and is not in the language.

(c) $\{w \in \Sigma^* \mid w \text{ starts and ends with the same character and also has an odd number of 1s}\}$.

3 A DFA with Four States (20 points; each part 10 points)

Consider the following DFA over the alphabet $\Sigma = \{a, b, c\}$:



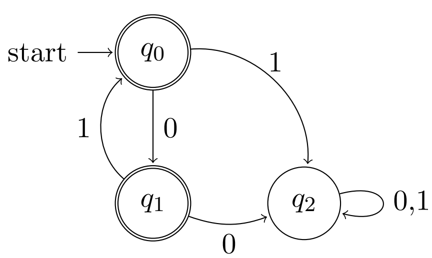
(a) Write out the formal definition of this DFA (as a 5-tuple). Describe the transition function in a table.

(b) What is the language recognized by this DFA? Give a short English description, then explain in a paragraph why your description is correct.

(c) (BONUS, 10 additional points) How many strings of length 32 are accepted by this DFA? Prove your answer.

4 Forwards and Backwards (20 points; each part 10 points)

Consider the following DFA over the alphabet $\Sigma = \{0, 1\}$:



(a) What is the language recognized by this DFA? Give a short English description, then explain in a paragraph why your description is correct.

(b) For a string w, let w^R denote the *reverse* of w, i.e., the same string written in reverse order. For instance, if $w = 11101010$ then $w^R = 01010111$. Let L denote the language

$$L = \{w \in \Sigma^* \mid w^R \text{ is accepted by the DFA from part a}\}.$$

Give a DFA for L and explain why it works. Try to use as few states as possible.

5 Languages With Short Strings (20 points; each part 10 points)

Let S denote the set of all languages L over the alphabet $\Sigma = \{0, 1\}$ which only contain strings of length exactly 4. (The languages in S may contain all the strings of length 4, just some of them, or no strings at all!)

(a) How many languages are in S? Give an explanation or proof for your answer.

(b) Prove that every language in S has a DFA that recognizes it.

(c) (BONUS, 10 additional points) For each $L \in S$, let $s(L)$ denote the minimum number of states needed by a DFA that recognizes L. What is the maximum, over all $L \in S$, of $s(L)$? Prove your answer. (Partial credit will be given for bounds close to the answer.)

6 Day-to-day DFA (9 points)

Give an example of a simple mechanical device you encounter in your daily life whose behavior could be modeled using a DFA. Explain the DFA and why it captures the behavior of the device. (Please use an example that's not in the textbook. If it helps, you can include a picture of the device.)

7 Collaborators and References (1 point)

Please list who you collaborated with on each problem, including any TAs or students you discussed the problems with in office hours. Also list any reference materials consulted other than the lectures and textbook for our class.