CS Theory (Spring '24) Assigned: March 18, 2024

Homework 4

Instructor: *Josh Alman* Due: March 27, 2024 at 11:59 pm

Collaboration on homework is allowed. However, you must write up solutions by yourself and understand everything that you hand in. You may also consult other reference materials, but you may not seek out answers from other sources.

Write the solution to each problem on a separate page. This includes writing each part of each problem on a separate page. Do not write your name or any other identifying information on any page of the submission; we are using anonymous grading in GradeScope.

Be sure to answer the final question listing your collaborators and other sources used. See the course webpage for more details.

There are 120 total possible points, and also bonus questions worth an additional 20 points.

1 The mandatory CFG problem (45 points; each part 15 points)

Consider the Context-Free Grammar G over the alphabet $\Sigma = \{0, 1\}$ with starting variable A and the following production rules:

$$A \to 00A1$$
$$A \to 1A00$$
$$A \to B$$
$$B \to 11B0$$
$$B \to 0B11$$
$$B \to C$$
$$C \to 1C0$$
$$C \to 0C1$$
$$C \to \varepsilon$$

(a) Prove that if $w \in \{0, 1\}^*$ can be generated by G, then w has at most twice as many 0s as 1s.

(b) Give a string which can be generated by G in at least two different ways, and give two derivations for it.

(c) Give, with proof, a string of length 12 which can be generated by G in exactly one way.

(d) (BONUS, 10 additional points) Prove or disprove the following claim: If $w \in \{0, 1\}^*$ can be generated by G in a way which uses each of the production rules at least once, then there is exactly one way to generate w.

2 Closure Under Complement (20 points; each part 10 points)

(a) Prove that if L is a language over $\Sigma = \{0, 1\}$, and L is Turing-decidable, then the complement of L (i.e., the language $\bar{L} := \{0, 1\}^* \setminus L$) is Turing-decidable.

(b) Explain why your proof for part a does not work if "Turing-decidable" is replaced by "Turing-recognizable". (We'll see soon that recognizable languages are not closed under complement!)

3 Partition (15 points for part a)

Let L be a language over $\Sigma = \{0, 1, \#\}$ such that every $w \in L$ contains exactly one # character. Define the language $T(L)$ by:

$$T(L) := \{x \in \{0, 1\}^* \mid \text{ there is a } y \in \{0, 1\}^* \text{ such that } x\#y \in L\}.$$

(a) Prove that if L Turing-decidable, then $T(L)$ is Turing-recognizable.

(b) (BONUS, 10 additional points) Prove that if L is Turing-recognizable, then $T(L)$ is Turing-recognizable.

4 Ones (30 points)

Define the language L over $\Sigma = \{0, 1, \#\}$ by

$$L := \{x\#y \mid \ x, y \text{ are strings in } \{0, 1\}^* \text{ with the same number of 1s}\}.$$

Give the formal description of a Turing machine that decides L. Give it as a figure, similar to example 3.7/figure 3.8 and example 3.9/figure 3.10 from the textbook. You may use the same simplified notation as in those textbook examples. Explain what your machine is doing and why it's correct. (We'll have trouble grading your submission if we don't understand how your machine is supposed to work!)

5 March Madness (8 points)

It's March, which means it's time for the NCAA Division I women's basketball tournament! Most eyes will be on two teams:

- The Columbia Lions, who qualified for the NCAA tournament this year for the first time! They're led by Abbey Hsu, the highest-scoring player in Columbia basketball history. Their first tournament game is on Wednesday, March 20.
- The Iowa Hawkeyes, whose star Caitlin Clark is the all-time leading scorer from any school in NCAA Division I history for both men's and women's basketball.

Let L be the language over $\Sigma = \{0, 1\}$ defined as:

$$L = \begin{cases} \{0\}, & \text{if either the Lions or the Hawkeyes will win the tournament,} \\ \{1\}, & \text{otherwise.} \end{cases}$$

Determine (with proof) whether L is Turing-decidable.

6 Collaborators and References and Rules (2 points)

Do all three of the following:

- List who you collaborated with on each problem, including any TAs or students you discussed the problems with in office hours.
- List any reference materials consulted other than the lectures and textbook for our class.
- Follow the formatting rules listed at the beginning of the assignment!