Extensive form games $\longleftarrow$ "finest description"
$\downarrow$
Strategic form games
$\downarrow$
Coalition form games $\longleftarrow$ "coarsest description"

Coalition form games: data consists of a mapping that assigns to each coalition a payoff or set of payoffs that are attainable through some joint course of action. The analysis is typically axiomatic and solution concepts include core, stable sets, nucleolus, Shapley value

Strategic form games: data consists of a set of players, an action sets and payoff functions. Solution concepts include rationalizability, dominant strategy solutions, Nash equilibrium and its many refinements

Extensive form games: data consists of a directed graph that describes the temporal order of play, the actions and information that are available when a player moves, payoffs, etc. Solution concepts include all strategic form concepts plus concepts like subgame perfection and sequentiality.

Strategic form games (SFG):

Players: $N = \{1, .., n\}$

Strategies: $A_i \neq \varnothing$

Payoffs: $(a_1, .., a_n) \in A_1 \times \cdots \times A_n \mapsto g_i(a_1, .., a_n) \in \mathbb{R}$

Some Notation:

$A = A_1 \times \cdots \times A_n$

$A_{-i} = \times_{j \neq i} A_j$

If $a \in A$, then a_{-i} denotes the "projection" of a onto A_{-i}, i.e., $a_{-i} = (a_j)_{j \neq i}$

If $a_{-i} \in A_{-i}$, then $(a_{-i}, b_i) := (a_1, .., b_i, .., a_n) \in A_1 \times \cdots \times A_n$

If $a \in A$, then $(a_{-i}, a_i) = a$

Nash equilibrium: A strategy profile $(a_1^*, .., a_n^*) \in A$ is a *Nash equilibrium* for the SFG if

$$a_i^* \in \arg\max_{a_i \in A_i} g_i(a_{-i}^*, a_i) \text{ for each } i \in N.$$

Dominant Strategy equilibrum: : A strategy $\hat{a}_i \in A_i$ is a *dominant strategy* for i if

$$\hat{a}_i \in \arg\max_{a_i \in A_i} g_i(a_{-i}, a_i)$$

for each $a_{-i} \in A_{-i}$.

A strategy profile $(a_1^*, .., a_n^*) \in A$ is a *dominant strategy equilibrium* for the SFG if a_i^* is a dominant strategy for each i, i.e.,

$$a_i^* \in \arg\max_{a_i \in A_i} g_i(a_{-i}, a_i)$$

for each $i \in N$ and each $a_{-i} \in A_{-i}$.

Every DS equilibrium is a Nash quilibrium but the converse is false.

$N = \{1, 2\}$

$A_1 = \{T, B\}$

$A_2 = \{L, R\}$

	L	R
T	1,1	-1,1
B	1,-1	0,0

$(a_1, a_2) = (T, L)$ is a Nash equilibrium; neither T nor L is a dominant strategy.

$(a_1, a_2) = (B, R)$ is a Nash equilibrium; both B and R are dominant strategies.

Important Question: Is (T, L) more "robust" than (B, R)?

Answer: Not clear

Main Existence Result:

The classic finite-dimensional existence result goes back to the work of Nash and the proof relies on fixed point theory.

Theorem: Suppose that

(i) each A_i is a nonempty, convex, compact subset of $\mathbb{R}^{m_i}$

(ii) each function $a \mapsto g_i(a)$ is continuous on A

(ii) each function $a_i \mapsto g_i(a_{-i}, a_i)$ is quasiconcave on A_i for each $a_{-i} \in A_{-i}$.

Then the SFG has an equilibrium.

This result includes the existence of mixed strategy equilibria in finite games as a special case.

Adding incomplete information: The General Interdependent Values Model

Players: $N = \{1, .., n\}$

Actions: $A_i \neq \varnothing$

Types: $T_i \quad (T \equiv T_1 \times \cdots \times T_n, T_{-i} = \times_{j \neq i} T_j)$, assume finite only for simplicity of presentation

Common prior: $P \in \Delta_T$:= probability measures defined on T with $P(t) > 0$ for all $t \in T$

Payoffs: $(a, t) \in A \times T \mapsto g_i(a|t)) \in \mathbb{R}$

Strategy for player i: $\sigma_i : T_i \to A_i$

Notation: $\sigma_{-i}(t_{-i}) := (\sigma_j(t_j))_{j \neq i}$

Ex-post payoff to player i of type $t_i \in T_i$ who chooses action a_i when opponents with type profile t_{-i} choose the strategy profile $\sigma_{-i} = (\sigma_j)_{j \neq i}$:

$$U_i(\sigma_{-i}(t_{-i}), a_i | t_{-i}, t_i) := g_i(\sigma_1(t_1), .., a_i, .., \sigma_n(t_n) | t_{-i}, t_i)$$

The basic soluition for games with incomplete information was proposed by Harsanyi:

Bayes-Nash equilibrium: (Harsanyi) A strategy profile $(\sigma_1^*, .., \sigma_n^*) \in A$ is a *Bayes-Nash equilibrium* for the asymmetric information game with private values if

$$\sigma_i^*(t_i) \in \arg\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} U_i(\sigma_{-i}^*(t_{-i}), a_i | t_{-i}, t_i) P(t_{-i} | t_i)$$

for each $i \in N$ and each $t_i \in T_i$.

Ex Post Nash equilibrium: A strategy profile $(\sigma_1^*, .., \sigma_n^*) \in A$ is an *ex-post Nash equilibrium* if

$$\sigma_i^*(t_i) \in \arg\max_{a_i \in A_i} U_i(\sigma_{-i}^*(t_{-i}), a_i | t_{-i}, t_i)$$

for all $i \in N$, $t_i \in T_i$ and $t_{-i} \in T_{-i}$.

Ex Post Dominant Strategy equilibrium: A strategy profile $\overline{\sigma}_i$ is an *ex-post dominant strategy* for player i if

$$\overline{\sigma}_i(t_i) \in \arg\max_{a_i \in A_i} U_i(a_{-i}, a_i | t_{-i}, t_i)$$

for each $t_i \in T_i$ and each $a_{-i} \in A_{-i}$.

A strategy profile $(\sigma_1^*, .., \sigma_n^*) \in A$ is an *ex-post dominant strategy equilibrium* for the asymmetric information game with private values if σ_i^* is a dominant strategy for each i, i.e., if

$$\sigma_i^*(t_i) \in \arg\max_{a_i \in A_i} U_i(a_{-i}, a_i | t_{-i}, t_i)$$

for all $i \in N$, $t_i \in T_i, t_{-i} \in T_{-i}$, and $a_{-i} \in A_{-i}$.

From the definitions, we have the following taxonomy;

$$\begin{aligned} & \textbf{Ex Post DS equilibrium} \\ \Rightarrow\ & \textbf{Ex Post Nash equilibrium} \\ \Rightarrow\ & \textbf{Bayes – Nash equilibrium} \end{aligned}$$

The Basic Implementation Problem:

economic agents: $N = \{1, .., n\}$

types of agent i: T_i (finite)

common prior: $P \in \Delta_T^* :=$ probability measures defined on T with $P(t) > 0$ for all $t \in T$

set of social alternatives: C

valuation function of agent i : private values

$$v_i : C \times T_i \rightarrow \mathbb{R}_+$$

where $v_i(c, t_i)$ denotes payoff to an agent i of type t_i when the social outcome is $c \in C$.

valuation function of agent i : interdependent values

$$v_i : C \times T \rightarrow \mathbb{R}_+$$

where $v_i(c, t)$ denotes payoff to an agent i given type profile t when the social outcome is $c \in C$.

Choice Problems and Mechanisms:

social choice problem: a collection $(v_1, .., v_n, P)$ where $P \in \Delta_T$.

outcome function: a mapping $q : T \to C$ that specifies an outcome in C for each profile of announced types.

direct mechanism: a collection $(q, x_1, .., x_n)$ where

$$q : T \to C$$

is an outcome function and

$$x_i : T \to \mathbb{R}$$

is a *transfer function*.

Definition: Let $(v_1, .., v_n, P)$ be a social choice problem.

An outcome function is *outcome efficient* if for each $t \in T$,

$$q(t) \in \arg\max_{c \in C} \sum_{j \in N} v_j(c; t).$$

A payment system $(x_1, .., x_n)$ is *feasible* if for each $t \in T$,

$$\sum_{j \in N} x_j(t) \leq 0 \ .$$

Individual rationality

Let $(v_1, .., v_n, P)$ be a social choice problem.

A mechanism $(q, (x_i))$ is *ex post individually rational* for agent i if

$$U_i(t_{-i}, t_i | t_i) \geq 0 \text{ for all } (t_{-i}, t_i) \in T.$$

A mechanism $(q, (x_i))$ is *interim individually rational* for agent i if

$$\sum_{t_{-i} \in T_{-i}} U_i(t_{-i}, t_i | t_i) P(t_{-i} | t_i) \geq 0 \text{ for all } t_i \in T_i.$$

Obviously, ex post IR implies interim IR.

Incentive compatibility: The problem

In a world of complete information in which a benign decision maker knows the actual profile of types, then "implementation" of an efficient social outcome is simple:

Step 1: Given $t \in T$, compute $q(t) \in \arg\max_{c \in C} \sum_{j \in N} v_j(c;t)$

Step 2: Define $x_i(t) = -v_i(q(t);t)$ (you pay exactly your valuation)

Step 3: By repeating steps 1 and 2 for each $t \in T$, you have constructed a mechanism $(q,(x_i))$ that is feasible, outcome efficient and even ex post individually rational

Now suppose that the benign decision maker does not know the actual type profile but tells the agents that, upon hearing their announcements, he will choose a social outcome and monetary transfers according to the mechanism $(q,(x_i))$ constructed above. Will agents truthfully report their types? Typically, the answer is typically "no" for this particular mechanism.

Incentive Compatibility: Private Values

Abusing notation, let

$$v_i(c,t) = v_i(c,t_i)$$

A direct mechanism $(q,(x_i))$ induces a game of incomplete information in which the agents are the players. In this game, $A_i = T_i$ and a strategy is a map $\sigma_i : T_i \to T_i$, i.e., agent i reports $\sigma_i(t_i)$ to the decision maker when his true type is t_i.

The ex post payoff to agent i when i reports t_i' and true type profile is (t_{-i}, t_i) and the other agents use reporting strategies σ_{-i} is

$$\begin{aligned} & U_i(\sigma_{-i}(t_{-i}), t_i' | t_{-i}, t_i) \\ = \;& v_i(q(\sigma_{-i}(t_{-i}), t_i'); t_i) + x_i(\sigma_{-i}(t_{-i}), t_i'). \end{aligned}$$

If $\sigma_j(t_j) = t_j$ for all j, then this ex post payoff is simply

$$\begin{aligned} & U_i(t_{-i}, t_i' | t_{-i}, t_i) \\ = \;& v_i(q(t_{-i}, t_i'); t_i) + x_i(t_{-i}, t_i'). \end{aligned}$$

The goal of implementation theory: given a social choice rule q, find transfers (x_i) so that the associated direct mechanism $(q, (x_i))$ induces a game of incomplete information in which truthful reporting is an equilibrium, i.e., σ^* is an equilibrium where $\sigma_i^*(t_i) = t_i$ for all i.

Remarks:

• Note the indefinite article: we wish to construct a mechanism in which truthful reporting is **an** equilibrium. It will almost never be the unique equilibrium.

• We have several notions of equilibrium corresponding to varying degrees of robustness. These are discussed on the next slide.

Definition: Let $(v_1, .., v_n, P)$ be a social choice problem with private values.

A mechanism $(q, (x_i))$ is:

interim incentive compatible if truthful reporting is a *Bayes-Nash equilibrium*: for each $i \in N$ and all $t_i, t_i' \in T_i$

$$\sum_{t_{-i} \in T_{-i}} U_i(t_{-i}, t_i' | t_i) P(t_{-i} | t_i)$$
$$\leq \sum_{t_{-i} \in T_{-i}} U_i(t_{-i}, t_i | t_i) P(t_{-i} | t_i)$$

ex post incentive compatible if truthful reporting is an *ex post Nash equilibrium*: for all $i \in N$, all $t_i, t_i' \in T_i$ and all $t_{-i} \in T_{-i}$,

$$U_i(t_{-i}, t_i' | t_i) \leq U_i(t_{-i}, t_i | t_i).$$

Let q be an outcome efficient social choice function. The *Vickrey-Clark-Groves (pivotal) transfers* are defined as follows:

$$\alpha_i^q(t) = \sum_{j \in N \setminus i} v_j(q(t); t_j) - \max_{c \in C} \left[\sum_{j \in N \setminus i} v_j(c; t_j) \right]$$

for each $t \in T$. The resulting mechanism $(q, (\alpha_i^q))$ is the *VCG mechanism with private valuations*.

Remarks:

• Agents are assessed the cost that they impose on the remaining agents.

• It is straightforward to show that the VCG mechanism is ex post individually rational and feasible.

• Furthermore, the VCG mechanism is ex post incentive compatible: in the private values model, truthful reporting is an ex post Nash equilibrium. In the private values model, a dominant strategy. In fact, truthful reporting is actually an ex post DS equilibrium.

Special case of VCG Mechanisms: Second price auctions with private values

If i receives the object, his value is the nonnegative number $w_i(t_i)$.

In this framework,

$$q(t) = (q_1(t), .., q_n(t))$$

where each $q_i(t) \geq 0$ and $q_1(t) + \cdots + q_n(t) \leq 1$ and

$$\begin{aligned} & v_i(q(t_{-i}, t_i'); t_i) + x_i(t_{-i}, t_i') \\ = \;& q_i(t_{-i}, t_i')w_i(t_i) + x_i(t_{-i}, t_i'). \end{aligned}$$

Finally, outcome efficiency means that

$$\sum_{i \in N} q_i(t)w_i(t_i) = \max_{i \in N}\{w_i(t)\}.$$

Let

$$w^*(t) := \max_i \hat{w}_i(t_i)$$

$$I(t) = \{i \in N | \hat{w}_i(t_i) = w^*(t_i)\}$$

and, again for simplicity of presentation, suppose that $|I(t)| = 1$. If

$$\begin{aligned} q_i^*(t) &= 1 \text{ if } I(t) = \{i\} \\ &= 0 \text{ if } I(t) \neq \{i\} \end{aligned}$$

then q^* is outcome efficient. Defining

$$w_{-i}^*(t_{-i}) := \max_{j:j \neq i} \{w_j(t_j)\}$$

then the VCG transfers associated with q^* are given by

$$\begin{aligned} \alpha_i^*(t) &= -w_{-i}^*(t_{-i}) \text{ if } I(t) = \{i\} \\ &= 0 \text{ if } I(t) \neq \{i\}. \end{aligned}$$

Why does the second price auction induce truthful announcements in the private values case?

Suppose that (t_{-i}, t_i) the true type profile and t_i' is i's announcement. If i is the winner when he is honest (i.e., $I(t_{-i}, t_i) = \{i\}$), then honesty yields a payoff of

$$w_i(t_i) - w^*_{-i}(t_{-i}) \geq 0.$$

However, a lie (i.e., $t_i' \neq t_i$)) results in one of two possibilities: either $I(t_{-i}, t_i') = \{i\}$ in which case his payoff remains

$$w_i(t_i) - w^*_{-i}(t_{-i})$$

or i is a loser and his payoff is 0.

What about interdependent values?

Incentive Compatibility: Interdependent Values

Definition: Let $(v_1, .., v_n, P)$ be a social choice problem. A mechanism $(q, (x_i))$ is:

interim incentive compatible if truthful revelation is a *Bayes-Nash equilibrium*: for each $i \in N$ and all $t_i, t_i' \in T_i$

$$\sum_{t_{-i} \in T_{-i}} U_i(t_{-i}, t_i' | t_i) P(t_{-i}, t_{-i} | t_i)$$
$$\leq \sum_{t_{-i} \in T_{-i}} U_i(t_{-i}, t_i | t_{-i}, t_i) P(t_{-i} | t_i)$$

ex post (Nash) incentive compatible if truthful revelation is an *ex post Nash equilibrium*: for all $i \in N$, all $t_i, t_i' \in T_i$ and all $t_{-i} \in T_{-i}$,

$$U_i(t_{-i}, t_i' | t_i) \leq U_i(t_{-i}, t_i | t_i).$$

A "super robust notion of incentive compatibility is possible. From the previous slide, recall that a mechanism $(q, (x_i))$ is:

ex post (Nash) incentive compatible if truthful revelation is an *ex post Nash equilibrium*: for all $i \in N$, all $t_i, t_i' \in T_i$ and all $t_{-i} \in T_{-i}$,

$$U_i(t_{-i}, t_i'|t_i) \leq U_i(t_{-i}, t_i|t_i).$$

A mechanism $(q, (x_i))$ is *ex post DS incentive compatible* if truthful revelation is an *ex post DS equilibrium*: for all $i \in N$, all $t_i, t_i' \in T_i$ and all $t_{-i}, s_{-i} \in T_{-i}$,

$$\begin{aligned} & v_i(q(s_{-i}, t_i'); t_{-i}, t_i) + x_i(s_{-i}, t_i') \\ \leq \;& v_i(q(s_{-i}, t_i); t_{-i}, t_i) + x_i(s_{-i}, t_i). \end{aligned}$$

Remarks:

• ex post DS incentive compatible $\Rightarrow$ ex post (Nash) incentive compatible $\Rightarrow$ interim incentive compatible

• ex post DS incentive compatibility is simply too strong to be useful

• good news: in the private values special case, ex post DS incentive compatibility and ex post (Nash) incentive compatibility coincide and reduce to the previous definition of DS incentive compatibility provided above for the private values case.

What about extending the VCG mechanism to the case of interdependent values?

Let q be an outcome efficient social choice function.

The *Generalized Vickrey-Clark-Groves (pivotal) transfers* are defined as follows: for each $t \in T$,

$$\alpha_i^q(t) = \sum_{j \in N \setminus i} v_j(q(t); t) - \max_{c \in C} \left[\sum_{j \in N \setminus i} v_j(c; t) \right]$$

The resulting mechanism $(q, (\alpha_i^q))$ is the *GVCG mechanism with interdependent valuations*.

Remarks:

• Agents are again assessed the cost that they impose on the remaining agents.

• It is straightforward to show that the GVCG mechanism is ex post individually rational and feasible.

• If v_i depends only on t_i, then the GVCG mechanism reduces to the classical VCG mechanism for private value problems

• In general, however, the GVCG mechanism will *not even* satisfy interim IC.

Question: Are there any circumstances under which the GVCG mechanism will be ex post IC?

Answer: Yes

An application of Implementation Theory with Interdependent Valuations: Cybersecurity

Agents: $N =$ set of nodes/users of a computer network

Node i's type: t_i is a measure of i's "vulnerability", e.g., i's location in the network, i's attractiveness to attackers, i's current level of security resource

$q_i =$ quantity of resource allocated to security at node i

$v_i(q_i, t) = i$'s willingness to pay for q_i units of the security resource given vulnerability profile $t = (t_1, .., t_n)$

Not unreasonable assumptions:

$$
\begin{aligned}
\frac{\partial v_i}{\partial x} &\geq 0 \\
\frac{\partial v_i}{\partial t_i} &\geq 0 \\
\frac{\partial v_i}{\partial t_j} &\geq 0, j \neq i \\
\frac{\partial v_i}{\partial t_i} &\geq \frac{\partial v_i}{\partial t_j}, j \neq i
\end{aligned}
$$

Suppose that

$$(q_1(t), .., q_n(t_n)) \in \arg \min_{(c_1,..,c_n)\in C} \sum_i v_i(c_i, t_1, .., t_n)$$

Now find a system of transfers $x = (x_1, .., x_n)$ so that (q, x) is IC and IR. For this implementation problem with interdep valuations, we can define the GVCG mechanism but it has desirable incentive properties only in special circumstances.

A more realistic cybersecurity model

Given a security resource allocation $(q_1, .., q_n)$ and a vulnerability profile $(t_1, .., t_n)$, let

$$G(q_1, .., q_n, t_1, .., t_n)$$

denote the expected monetary value of the damage incurred by the **network** if a node is attacked.

Example: Given $(q_1, .., q_n)$ and $(t_1, .., t_n)$, the network administrator knows/estimates that a malevolent agent will attack node i with prob $\pi_i(q_1, .., q_n, t_1, .., t_n)$

If node i is attacked, then $w_i(q_i, t_1, .., t_n) =$ expected value of **network damage** incurred.

In this case,

$$\begin{aligned} & G(q_1, .., q_n, t_1, .., t_n) \\ = & \sum_i \pi_i(q_1, .., q_n, t_1, .., t_n) w_i(q_i, t_1, .., t_n) \end{aligned}$$

As a further specialization, suppose that

$$\pi_i(q_1, .., q_n, t_1, .., t_n) = \pi_i(q_i, t_1, .., t_n)$$

and

$$v_i(q_i, t) = -\pi_i(q_i, t_1, .., t_n) w_i(q_i, t_1, .., t_n).$$

Then the incentives of the users and the system administrator are "aligned" but this is not a realistic assumption.

Reasonable assumptions:

$$\begin{array}{rcl} t_i & \mapsto & \pi_i(q_1,..,q_n,t_1,..,t_n) \text{ is increasing} \\ t_j & \mapsto & \pi_i(q_1,..,q_n,t_1,..,t_n) \text{ is decreasing}, \quad j \neq i \\ q_i & \mapsto & \pi_i(q_1,..,q_n,t_1,..,t_n) \text{ is decreasing} \\ q_i & \mapsto & w_i(q_i,t_{-i},t_i) \text{ is decreasing} \\ t_i & \mapsto & w_i(q_i,t_{-i},t_i) \text{ is increasing} \end{array}$$

If

$$q(t) \in \arg\min_q G(q_1,..,q_n,t_1,..,t_n)$$

then we want to find a system of transfers $x = (x_1,..,x_n)$ so that (q,x) is IC and IR.

For this implementation problem with interdependent valuations, the GVCG mechanism is generally irrelevant even in the case of private values in which $v_i(x,t) = v_i(x,t_i)$!!

So we need transfers different from the VCG/GVCG if we want to implement the outcome function q.

The "typical" mechanism design with a continuum of types: The case of independent private values

Types: $t_i \in T_i = [a_i, b_i]$

Probabilistic structure: f_i := density function for the distribution of agent i's type

$$t = (t_1, .., t_n) \in T \mapsto f(t) = f_1(t_1) \cdots f_1(t_1)$$

$$t_{-i} \in T_{-i} \mapsto f_{-i}(t_{-i}) = \prod_{j \neq i} f_j(t_j)$$

$$\text{maximize} \int_T H(x(t), q(t), t) f(t) dt$$

$$t_i \in \arg\max_{t_i'} \int_{T_{-i}} \left[v_i(q(t_{-i}, t_i'); t_i) - x_i(t_{-i}, t_i') \right] f_{-i}(t_{-i}) dt_{-i}$$

$$\int_{T_{-i}} \left[v_i(q(t_{-i}, t_i); t_i) - x_i(t_{-i}, t_i) \right] f_{-i}(t_{-i}) dt_{-i} \text{ for all } i, t_i$$

Example: The Optimal Auction Design Problem (Myerson, MOR, 1980)

An outcome is profile of probabilities denoted $q(t) = (q_1(t), .., q_n(t))$ and i's ex post expected payoff is:

$$v_i(q(t_{-i}, t_i'); t_i) - x_i(t_{-i}, t_i') = q_i(t_{-i}, t_i')t_i - x_i(t_{-i}, t_i')$$

The seller then chooses a mechanism (q, x) that solves *Problem A*:

$$\text{maximize} \int_T \left[\sum_i x_i(t) \right] f(t)dt$$

$$t_i \in \arg\max_{t_i'} \int_{T_{-i}} \left[q_i(t_{-i}, t_i')w_i(t_i) - x_i(t_{-i}, t_i') \right] f_{-i}(t_{-i})dt_{-i}$$

$$\int_{T_{-i}} \left[q_i(t_{-i}, t_i')w_i(t_i) - x_i(t_{-i}, t_i) \right] f_{-i}(t_{-i})dt_{-i} \text{ for all } i, t_i$$

$$q_i(t) \geq 0 \text{ and } q_1(t) + \cdots + q_n(t) \leq 1$$

Example: The Optimal Combinatorial Auction Design Problem (Ulku, 2009)

Let Ω denote a finite set of objects. An outcome is profile of subsets of Ω denoted

$$S(t) = (S_1(t), .., S_n(t))$$

and i's ex post expected payoff is:

$$v_i(S(t_{-i}, t_i'); t_i) - x_i(t_{-i}, t_i')$$

The seller then chooses a mechanism (S, x) that solves

$$\text{maximize} \int_T \left[\sum_i x_i(t) \right] f(t)dt$$

$$t_i \in \arg\max_{t_i'} \int_{T_{-i}} \left[v_i(S(t_{-i}, t_i'); t_i) - x_i(t_{-i}, t_i') \right] f_{-i}(t_{-i})dt_{-i}$$

$$\int_{T_{-i}} \left[v_i(S(t_{-i}, t_i'); t_i) - x_i(t_{-i}, t_i) \right] f_{-i}(t_{-i})dt_{-i} \text{ for all } i, t_i$$

$$S_i(t) \cap S_j(t) = \varnothing, i \neq j \text{ and } S_1(t) \cup \cdots \cup S_n(t) \subseteq \Omega$$

Example of a combinatorial auction design problem: Click Auctions

$\Omega = \{1, ..., m\}$ be the set of ad positions that will be displayed by an internet search engine after a keyword search ranked from top to bottom.

$k \in \Omega \mapsto \alpha_k$ interpreted as the number of user clicks on the ad displayed at that position.

Suppose that the positions $1, .., m$ are ranked according to "clicks per unit time" so that $\alpha_1 > \cdots > \alpha_m$.

Let $N = \{1, ..., n\}$ be the set of potential advertisers.

The value of position k to advertiser i of type t_i who is assigned position k is defined as $g_i(\alpha_k, t)$ where $k < j$ implies $g_i(\alpha_k, t_i) > g_i(\alpha_j, t_i)$.

Possible valuations for sets of positions:

$$\begin{aligned} v_i(S, t_i) &= \max_{k \in S}\{g_i(\alpha_k, t_i)\} \\ v_i(S, t_i) &= \sum_{k \in S} g_i(\alpha_k, t_i) \end{aligned}$$

An "atypical" mechanism design with a continuum of types: Cybersecurity

Agents: $N =$ set of nodes/users of a computer network

Node i's type: t_i is a measure of i's "vulnerability", e.g., i's location in the network, i's attractiveness to attackers, i's current level of security resource

$q_i =$ quantity of resource allocated to security at node i

$v_i(x, t) = i$'s willingness to pay for x units of the security resource given vulnerability profile $t = (t_1, .., t_n)$

$G(q_1, .., q_n, t_1, .., t_n) =$ expected monetary value of the damage incurred by the **network** given security resource allocation $(q_1, .., q_n)$ and vulnerability profile $(t_1, .., t_n)$.

Model 1: Suppose that $0 < \theta \leq 1$.

$$\text{minimize} \int_T \left[\theta G(q(t),t) + (1-\theta)C(q(t))\right] f(t)dt$$
$$IC, IR$$
$$\sum_i x_i(t) \geq C(q(t)) \text{ for all } t$$

Model 2: Suppose that $0 < \alpha < 1$.

$$\text{minimize} \int_T C(q(t)) f(t)dt$$
$$IC, IR$$
$$G(q(t),t) \leq \alpha \text{ for all } t$$
$$\sum_i x_i(t) \geq C(q(t)) \text{ for all } t$$

How do we solve the "basic" optimal auction design problem? Myerson proved the following fundamental result:

Theorem: If q^* solves the problem

$$\text{maximize} \int_T \left[\sum_i \left(t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \right) q_i(t_i) \right] f(t)dt \text{ s.t.}$$

$$q_i(t) \geq 0 \text{ and } q_1(t) + \cdots + q_n(t) \leq 1$$

$$t_i \mapsto \int_{T_{-i}} q_i(t_{-i}, t_i) f_{-i}(t_{-i}) dt_{-i} \text{ is nondecreasing for all } i, t_i$$

and if

$$x_i^*(t) = q_i^*(t)t_i - \int_{a_i}^{t_i} q_i^*(t_{-i}, y)dy$$

then (q^*, x^*) solves *Problem A*.

An observation: If q^* solves *Problem B*

$$\text{maximize} \int_T \left[\sum_i \left(t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \right) q_i(t_i) \right] f(t) dt \text{ s.t.}$$

$$q_i(t) \geq 0 \text{ and } q_1(t) + \cdots + q_n(t) \leq 1$$

and if

$$t_i \mapsto \int_{T_{-i}} q_i^*(t_{-i}, t_i) f_{-i}(t_{-i}) dt_{-i} \text{ is nondecreasing for all } i, t_i$$

then q^* solves *Problem A*.

Problem B is simple to solve. Choose $t \in T$.

If $\max_i \left\{ t_i - \frac{1-F_i(t_i)}{f_i(t_i)} \right\} \leq 0$, let $q_i^*(t) = 0$ for all t.

If $\max_i \left\{ t_i - \frac{1-F_i(t_i)}{f_i(t_i)} \right\} > 0$, let

$$i^*(t) \in \arg\max_i \left\{ t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \right\}$$

and let

$$q_{i^*(t)}(t) = 1.$$

Of course, this simple solution may not solve the real mechanism design problem unless we know that

$t_i \mapsto \int_{T_{-i}} q_i^*(t_{-i}, t_i) f_{-i}(t_{-i}) dt_{-i}$ is nondecreasing for all i, t_i