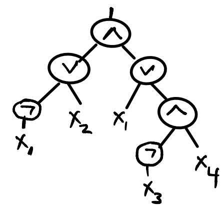
GO ASK alice!

ADVANCES AND NEW DIRECTIONS IN COMMUNICATION COMPLEXITY

Toniann Pitassi

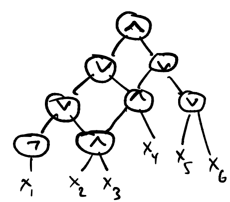
GIVEN A COMPUTATIONAL MODEL $\mathcal{M}$, HOW HARD IS IT TO COMPUTE A PARTICULAR FUNCTION $F:\{0,1\}^n \to \{0,1\}$?

$\mathcal{M}$ = Boolean formulas



Best LB: $4.5n$ [LR]

$\mathcal{M}$ = Boolean circuits



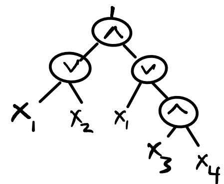
Best LB: $n^{3-o(1)}$ [Has'98]

$\mathcal{M}$ = AC^0_d circuits

Best LB: $2^{\Omega(n^{\frac{1}{d}})}$ [Has '86]

GIVEN A COMPUTATIONAL MODEL $\mathcal{M}$, HOW HARD IS IT TO COMPUTE A PARTICULAR FUNCTION $F: \{0,1\}^n \to \{0,1\}$?

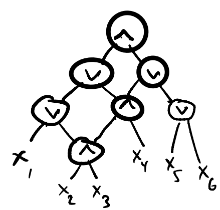
$\mathcal{M}$ = monotone Boolean formulas



Best LB: $2^{\Omega(n)}$

[PR'17]

$\mathcal{M}$ = monotone Boolean circuits



Best LB: $2^{n^{1/2-o(1)}}$

[CKR'20]

TODAY: unified approach to many lower bounds via COMMUNICATION COMPLEXITY

- Monotone formula Lower Bounds
- Monotone circuit Lower Bounds
- Monotone span program Lower Bounds
- Linear Program Extension complexity
- SDP Extension complexity

Theme: Complexity closely connected to proof complexity

COMMUNICATION COMPLEXITY (Yao '79)

10111

00110

$$F : \{0,1\}^n \times \{0,1\}^n \rightarrow \mathcal{O}$$

COMMUNICATION COMPLEXITY (Yao '79)

10111

00110

I don't think..

⋮

$F : \{0,1\}^n \times \{0,1\}^n \rightarrow 0$

COMMUNICATION COMPLEXITY (Yao '79)

10111

00110

I don't think..

Then don't talk

⋮

$$F : \{0,1\}^n \times \{0,1\}^n \to O$$

$$CC(F) = \min_{\pi} \; CC(\pi)$$

COMMUNICATION FOR SEARCH PROBLEMS

10111

00110

Example. (KW Search) $f : \{0,1\}^n \to \{0,1\}$

KW(f): Alice: $x \in f^{-1}(1)$ Bob: $y \in f^{-1}(0)$

Output $i \in [n]$ such that $x_i \neq y_i$

COMMUNICATION FOR SEARCH PROBLEMS

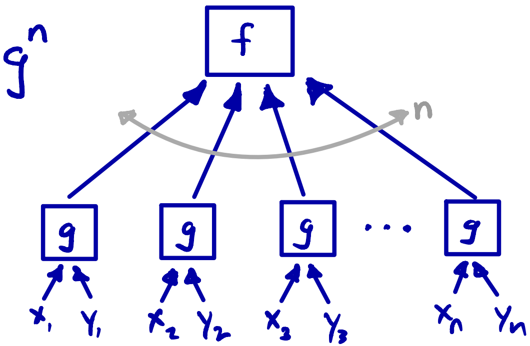
Example. (KW Search) $f : \{0,1\}^n \to \{0,1\}$, f monotone

mKW(f): Alice: $x \in f^{-1}(1)$ Bob: $y \in f^{-1}(0)$

Output $i \in [n]$ such that $x_i > y_i$

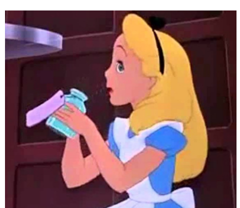
QUERY TO COMMUNICATION LIFTING

$f : \{0,1\}^n \to O \rightsquigarrow F = f \circ g^n$



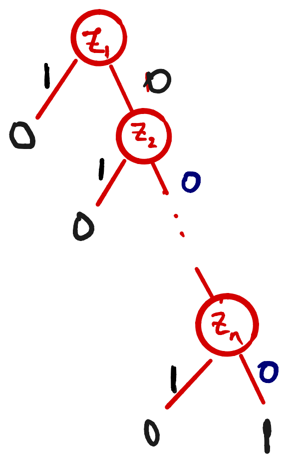
LIFTING THEOREM

Communication Complexity of F $\approx$ Query complexity of f

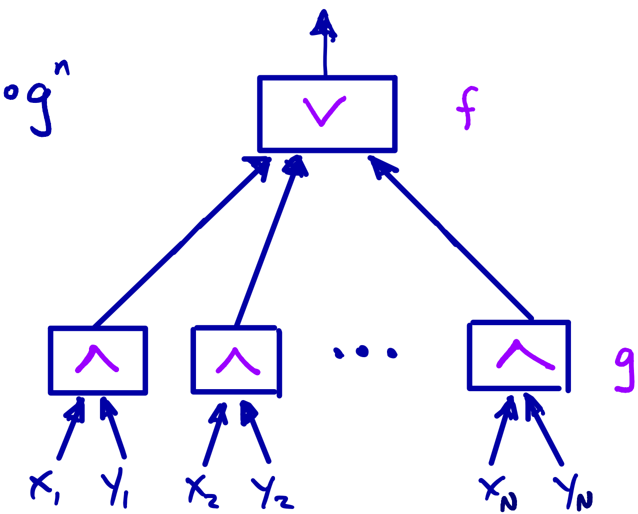


EXAMPLE: SET DISJOINTNESS

f:



$F = f \circ g^n$



$DT(f) = \Theta(n)$

$CC(f \circ g^n) = \Theta(n)$

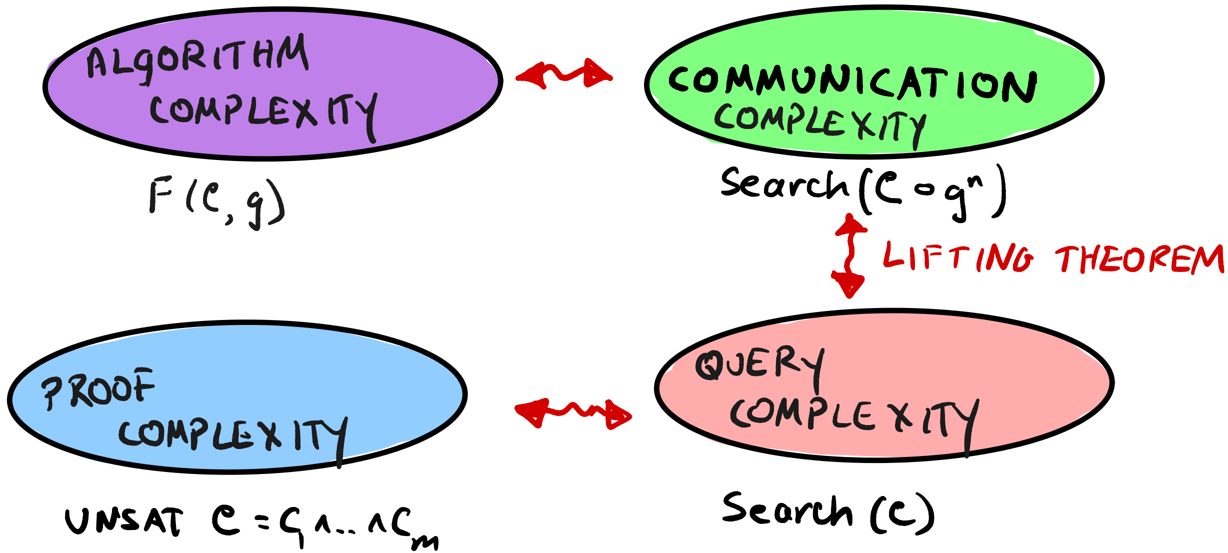
SOME LIFTING THEOREMS

QUERY MODEL	COMMUNICATION MODEL	REFERENCE
Decision Tree	Deterministic CC	[RM'99, CKLM'19]
Nondet DT	Nondeterministic CC	[GLM'16]
Randomized DT	Randomized CC	[GPW'17, CFKMP'21]
Polynomial Degree	Rank	[Sherstov'11, SZ'09, RS'10]
Decision List	Rectangle List	[GKPW'17]
Resolution/Dag-like DT	DAG-LIKE Communication	[GGKS'18]
Nullstellensatz degree	Algebraic Tiling	[RPCR'16, PR'17, PR'18]
Sherali Adams degree	Non-Negative Rank	[CLRS'16]
SOS degree	PSD Rank	[LRS'15]

APPLICATIONS OF LIFTING

1. Monotone formula/circuit Lower Bounds
2. Proof complexity
3. Cryptography (Lower Bounds for Secret Sharing Schemes)
4. Game Theory (Nash Equilibrium)
5. Graph Theory (Alon-Saks-Seymour Conjecture)
6. Linear/SDP Extended Formulations
7. Communication complexity Separations
8. Quantum complexity
9. Data Structures

LOWER BOUND PROGRAM



LOWER BOUND PROGRAM

ALGORITHMS	COMMUNICATION	QUERY	PROOFS	REFs
mFormulas	P^{cc}	P^{DT}	tree-Resolution	RM'99 GPW'18
mCircuits	PLS^{cc}	PLS^{DT}	Resolution	GGKS'18
mFormulas w/ Errors	BPP^{cc}	BPP^{DT}	Random Resolution	GPW'17
mSpan Programs	PPA^{cc}	PPA^{DT}	Nullsatz	PR'17 PR'18
LP Extension Complexity	$Rank^{+}$		SA	CLRS'16
SDP Extension Complexity	SDP Rank		SOS	LRS'15

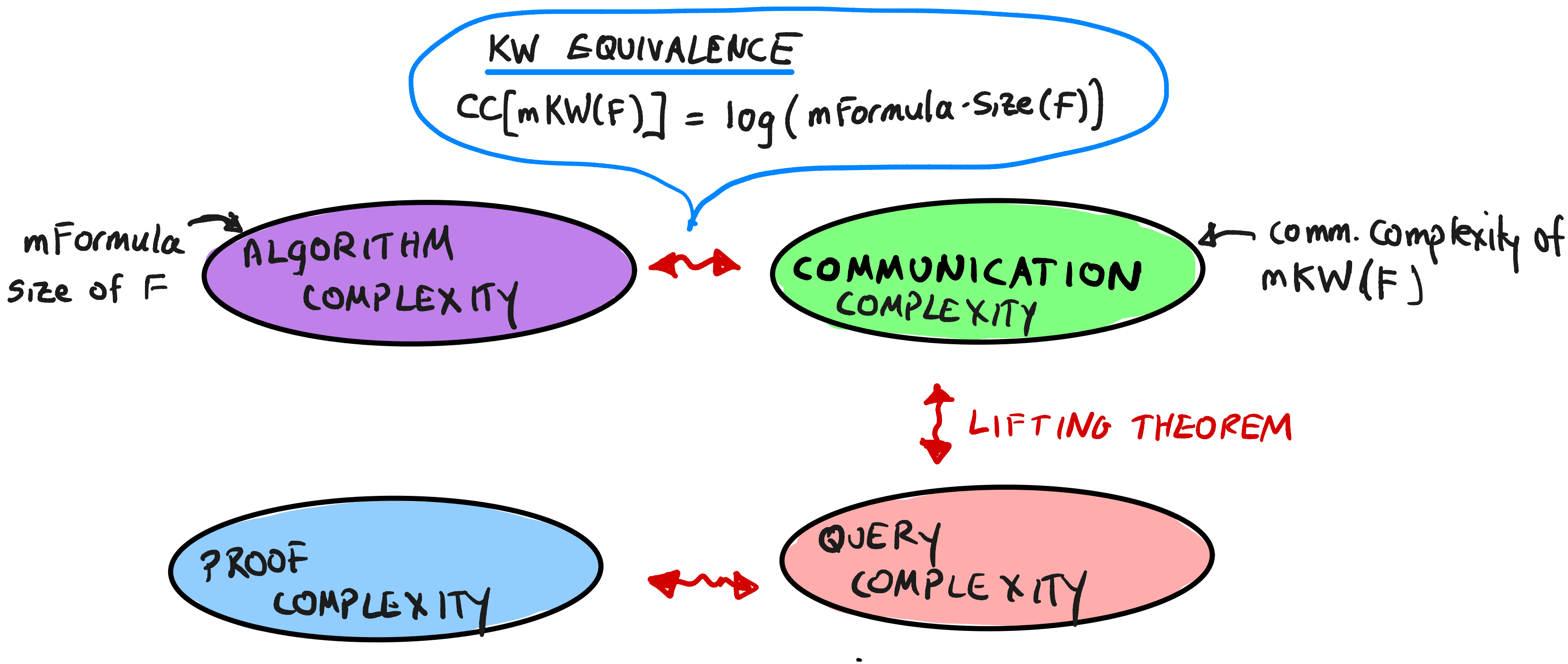
$$\text{Algorithm-Size}(F(\mathcal{C}, g)) \approx \text{Proof-size}(\mathcal{C})$$

LOWER BOUND PROGRAM

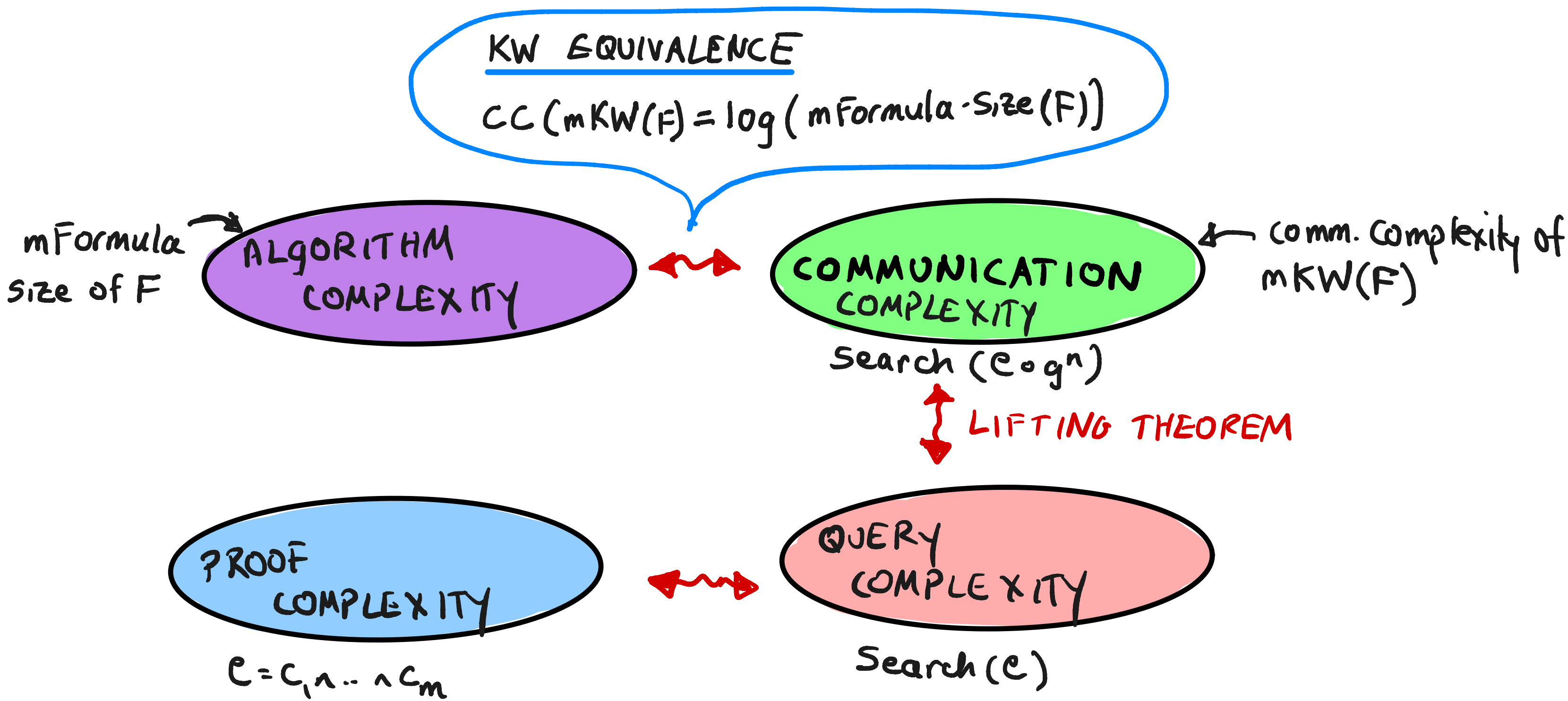
ALGORITHMS	COMMUNICATION	QUERY	PROOFS	REFs
mFormulas	P^{cc}	P^{DT}	tree-Resolution	RM'99 GPW'18
mCircuits	PLS^{cc}	PLS^{DT}	Resolution	GGKS'18
mFormulas w/ Errors	BPP^{cc}	BPP^{DT}	Random Resolution	GPW'17
mSpan Programs	PPA^{cc}	PPA^{DT}	Nullsatz	PR'17 PR'18
LP Extension Complexity	$Rank^{+}$		SA	CLRS'16
SDP Extension Complexity	SDP Rank		SOS	LRS'15

$$\text{Algorithm-Size}(F(c,g)) \approx \text{Proof-size}(c)$$

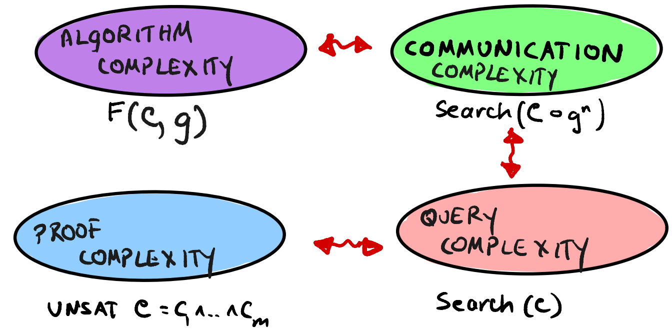
INSTANTIATION I : monotone Formula Size Lower Bounds



INSTANTIATION I : monotone Formula Size Lower Bounds



HARD-TO-REFUTE CNFs → HARD FUNCTIONS

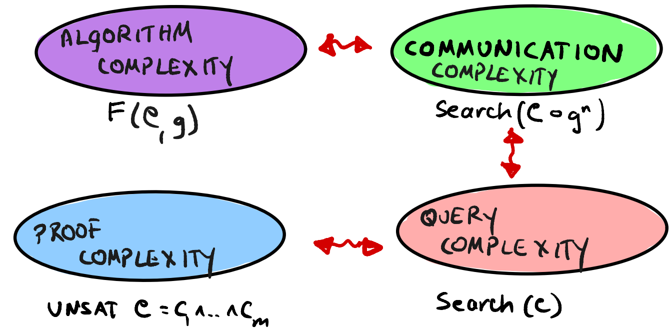


$\mathcal{C} = C_1 \wedge C_2 \wedge \dots \wedge C_m$ UNSAT CNF OVER $z_1 \dots z_n$

$\mathrm{Search}(\mathcal{C}) \subseteq \{0,1\}^n \times [m]$

given assignment γ to $z_1 \dots z_n$
output a falsified clause

HARD-TO-REFUTE CNFs $\to$ HARD FUNCTIONS



$\mathcal{C} = C_1 \wedge C_2 \wedge \dots \wedge C_m$ UNSAT CNF OVER $z_1 \dots z_n$

$\mathcal{C} \circ g^n$: CSP over $x_1 \dots x_n$ $y_1 \dots y_n$ where $z_i \leftarrow g(x_i, y_i)$

$\text{Search}(\mathcal{C}) \subseteq \{0,1\}^n \times [m]$ $\rightsquigarrow$ $\text{Search}(\mathcal{C} \circ g^n) \subseteq \mathcal{X}^n \times \mathcal{Y}^n \times [m]$

given assignment γ to $z_1 \dots z_n$ output a falsified clause

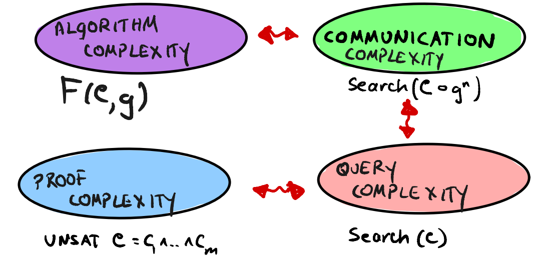
Alice: $x_1 \dots x_N$ Bob: $y_1 \dots y_N$

given assignment α to $x_1 \dots x_n$, β to $y_1, \dots, y_n$ output a falsified constraint of $\mathcal{C} \circ g^n$

Example

$\mathcal{C} = (z_1 \vee z_2)(\bar{z}_1 \vee z_3)$ $\longrightarrow$ $\mathcal{C} \circ \wedge^n = (x_1 y_1 \vee x_2 y_2)(\overline{x_1 y_1} \vee x_3 y_3)$

HARD-TO-REFUTE CNFs → HARD FUNCTIONS



$\mathcal{C} = C_1 \wedge C_2 \wedge \dots \wedge C_m$: UNSAT CNF OVER $z_1 \dots z_n$

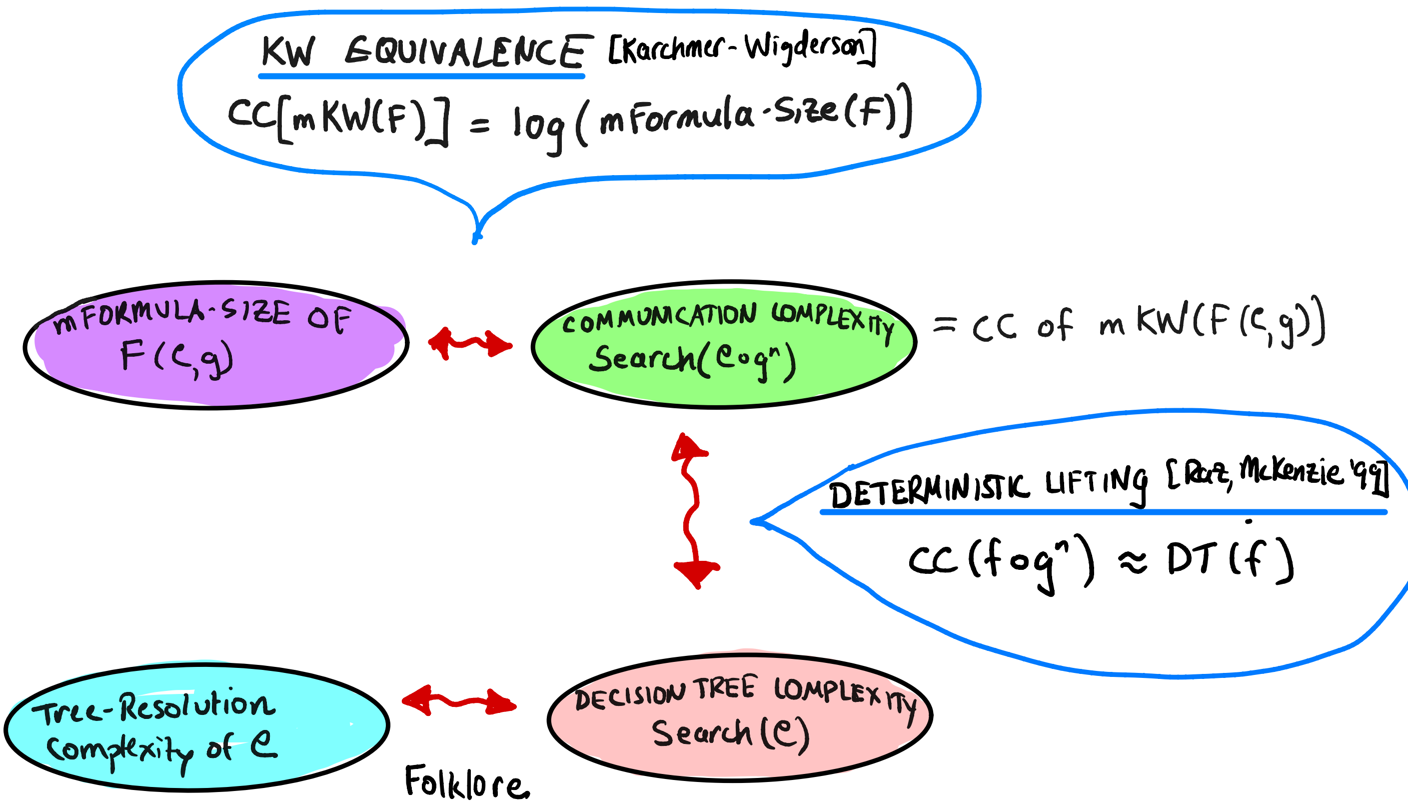
$\mathcal{C} \circ g^n$: CSP over $x_1 \dots x_n$ $y_1 \dots y_n$ where $z_i \leftarrow g(x_i, y_i)$

$\mathrm{Search}(\mathcal{C}\circ g^n) \subseteq \mathcal{X}^n \times \mathcal{Y}^n \times [m]$ $\rightsquigarrow$ $F(\mathcal{C},g) : \{0,1\}^{\mathrm{poly}(n)} \to \{0,1\}$

Theorem. For any unsat $\mathcal{C}$, there exists monotone $F = F(\mathcal{C}, g)$ such that $\mathrm{Search}(\mathcal{C}\circ g^n) \equiv \mathrm{mKW}(F)$

mKW_F: Alice: $x \in F^{-1}(1)$ Bob: $y \in F^{-1}(0)$
FIND i SUCH THAT $x_i > y_i$

INSTANTIATION I : mFORMULA-SIZE LOWER BOUNDS



INSTANTIATION II : mCIRCUIT-SIZE LOWER BOUNDS

KW EQUIVALENCE FOR CIRCUITS [Razborov]

$$\text{dag-CC}[KW(F)] = \text{mCircuitSize}(F)$$

mCIRCUIT-SIZE OF $F(\mathcal{C}, g)$ $\leftrightarrow$ DAG-LIKE CC of Search$(\mathcal{C} \circ g^n)$ $=$ dag-like CC of $mKW(F(\mathcal{C}, g))$

DAG-CC LIFTING [garg, göös, Kamath, Sokolov '18]

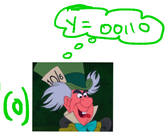
$$\text{dag-CC}(f \circ g^n) \approx \text{dag-DT}(f)$$

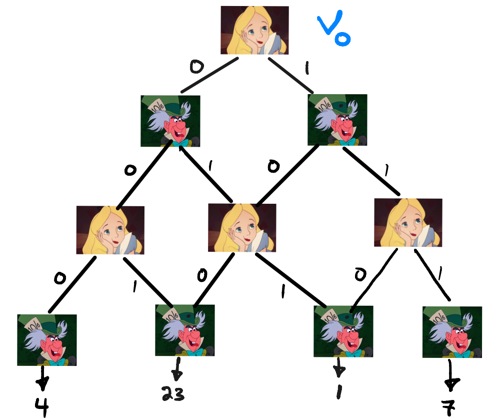
Tree-Resolution Complexity of $\mathcal{C}$ $\leftrightarrow$ DECISION TREE COMPLEXITY Search$(\mathcal{C})$

Folklore.

DAG-LIKE PROTOCOLS (PLS^{cc}) $\equiv$ mCircuit-Size [Razborov], [Sokolov]

mKW(F) : $x \in F^{-1}(1)$ 

$y \in F^{-1}(0)$ 



$$\Pi = (G = (V, E),\ v_0 \in V,\ \{R_v \mid v \in V\},\ \ell : V \to \mathcal{O}) :$$

Vertices labelled by Rectangles $\forall v \in V: R_v \subseteq \{0,1\}^n \times \{0,1\}$

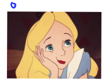
$R_{v_0} = \{0,1\}^n \times \{0,1\}^n$

Consistency If v has children v', v'' $\Rightarrow$ $R_v \subseteq R_{v'} \cup R_{v''}$

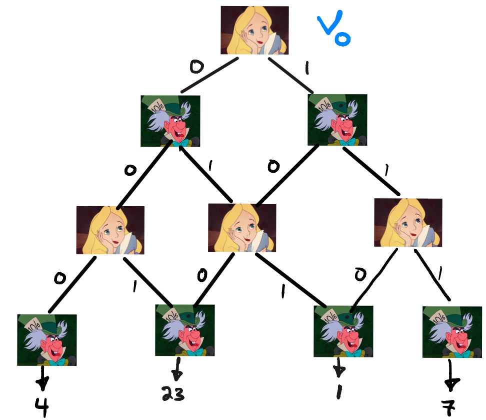
Correctness Each leaf vertex v has label $\ell(v) \in \mathcal{O}$ s.t. $\forall (x,y) \in R_v$ $(x, y, \ell(v)) \in \mathcal{O}$

DAG-LIKE PROTOCOLS (PLS^{cc}) $\equiv$ mCircuit-Size

$x = 10111$ $\qquad$ $y = 00110$

mKW(F): $x \in F^{-1}(1)$  $y \in F^{-1}(0)$ 

$\Pi = (\mathcal{G} = (V, E),\ v_0 \in V,\ \{R_v \mid v \in V\},\ \ell : V \to \mathcal{O})$:



Vertices labelled by Rectangles: $\forall v \in V:\ R_v \subseteq \{0,1\}^n \times \{0,1\}$

$R_{v_0} = \{0,1\}^n \times \{0,1\}^n$

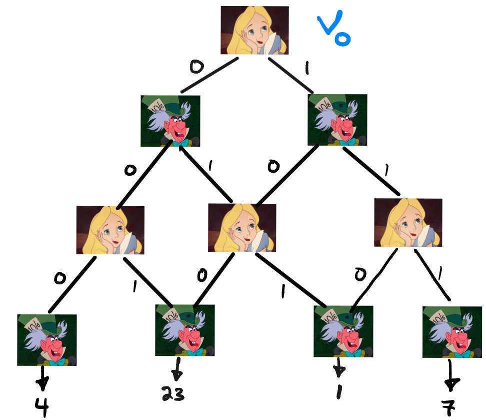
Consistency: If v has children $v', v'' \Rightarrow R_v \subseteq R_{v'} \cup R_{v''}$

Correctness: Each leaf vertex v has label $\ell(v) \in \mathcal{O}$ s.t. $\forall (x,y) \in R_v$ $(x, y, \ell(v)) \in \mathcal{O}$

Theorem $PLS^{cc}(KW_F)$ = CIRCUIT-SIZE(F)

DAG-LIKE PROTOCOLS (PLS^{cc}) $\equiv$ mCircuit-Size

$S \subseteq \{0,1\}^n \times \{0,1\}^n \times \mathcal{O}$



$\Pi = \left(\mathcal{G} = (V, E),\ v_0 \in V,\ \{R_v \mid v \in V\},\ \ell : V \to \mathcal{O} \right)$:

Vertices labelled by Rectangles: $\forall v \in V$: $R_v \subseteq \{0,1\}^n \times \{0,1\}^n$

$R_{v_0} = \{0,1\}^n \times \{0,1\}^n$

Consistency: If v has children $v', v'' \Rightarrow R_v \subseteq R_{v'} \cup R_{v''}$

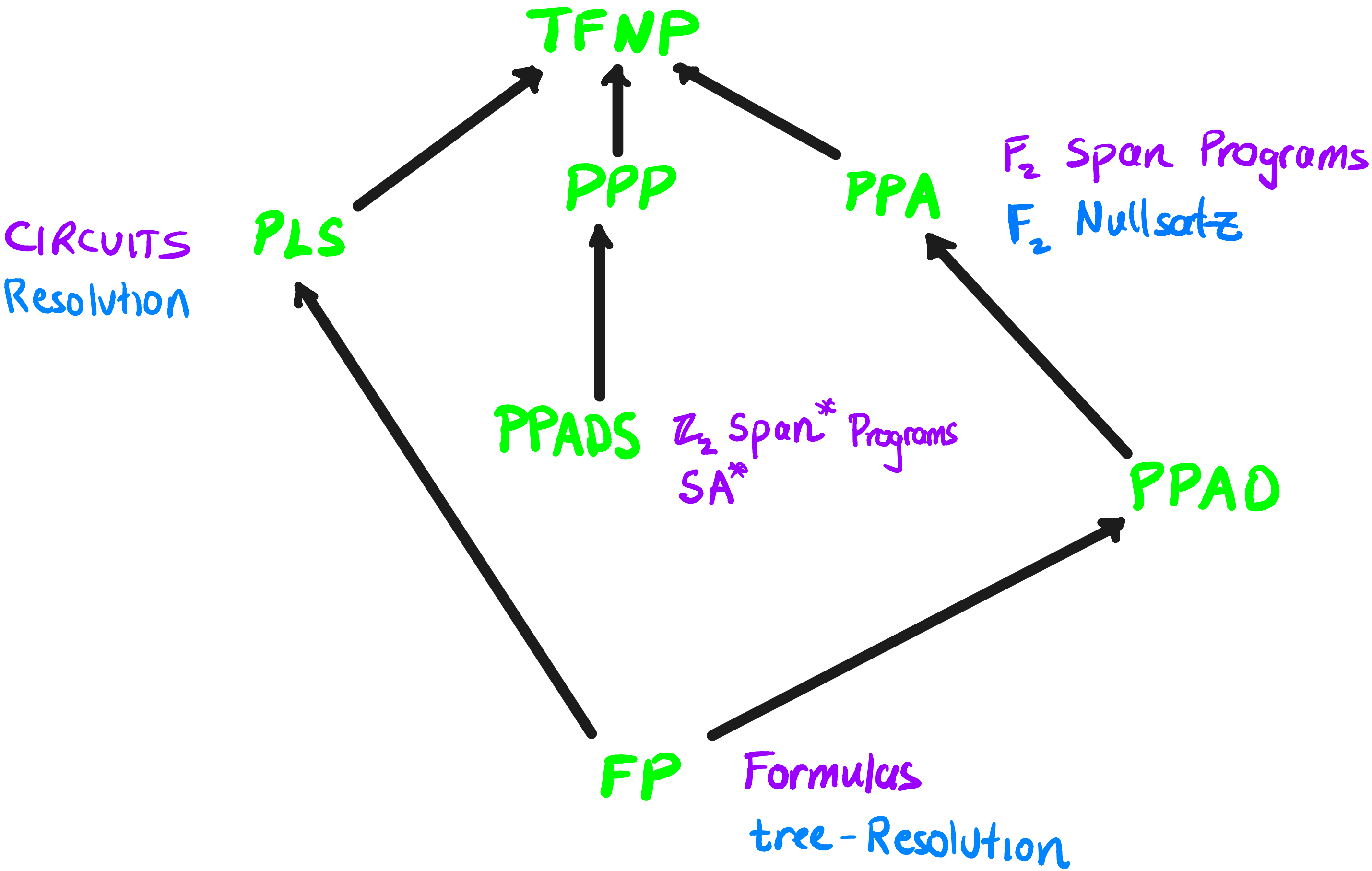
Correctness: Each leaf vertex v has label $\ell(v) \in \mathcal{O}$ s.t. $\forall (x,y) \in R_v$ $(x, y, \ell(v)) \in \mathcal{O}$

Theorem $PLS^{cc}(mKW_F) = mCIRCUIT\text{-}SIZE(F)$

LOWER BOUND PROGRAM

ALGORITHMS	COMMUNICATION	QUERY	PROOFS	REFs
mFormulas	P^{cc}	P^{DT}	tree-Resolution	RM'99 GPW'18
mCircuits	PLS^{cc}	PLS^{DT}	Resolution	GGKS'18
mFormulas w/ Errors	BPP^{cc}	BPP^{DT}	Random Resolution	GPW'17
mSpan Programs	PPA^{cc}	PPA^{DT}	Nullsatz	PR'17 PR'18
LP Extension Complexity	Rank^{+}		SA	CLRS'16
SDP Extension Complexity	SDP Rank		SOS	LRS'15

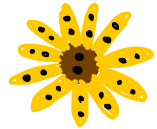
$$\text{Algorithm-Size}(F(C,g)) \approx \text{Proof-size}(C)$$


TFNP
PPP
PPA
$\mathbb{F}_2$ Span Programs
$\mathbb{F}_2$ Nullsatz
CIRCUITS
Resolution
PLS
PPADS
$\mathbb{Z}_2$ Span* Programs
SA*
PPAD
FP
Formulas
tree-Resolution

REST OF TALK . . .

- Lifting via Sunflowers
  and a web of interconnections

- New Applications/Directions
  and Some Open Problems

LIFTING VIA 

[Lovett, Meka, Mertz, P, Zhang '21]

f: n-bit boolean function / search problem

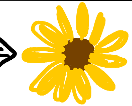
g: index gadget $\mathrm{IND}(x, y) = y_x$ $\quad |y| = \mathrm{poly}(n)$, $|x| = \log |y|$

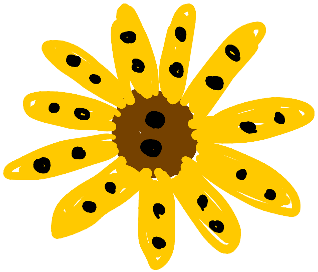
DETERMINISTIC LIFTING THEOREM [RM'99, GPW'17]

$$DT(f) \cdot \Theta(\log n) = CC(f \circ g^n)$$

- Uses Sunflower Lemma as black box
- Also holds for Dag-Like Lifting
- Improved gadget size $|y| = n^{1+\varepsilon}$

SUNFLOWER LEMMA

$|X|$ large $\Rightarrow$  $\in X$

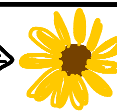


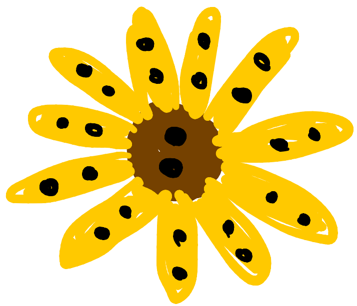
$n=4, \; p=11$

Let X be a n-uniform set system

If $|X| > r^n$ then X contains a sunflower with p petals

SUNFLOWER LEMMA

$|X|$ large $\Rightarrow$  $\in X$



$n=4$, $p=11$

Let X be a n-uniform set system

If $|X| > r^n$ then X contains a sunflower with p petals

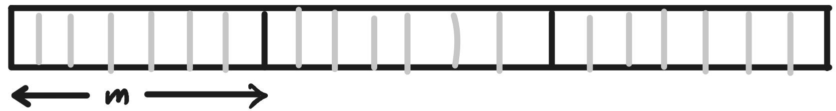
Old: True for $r \sim pn$

Conjecture: True for $r \sim p$

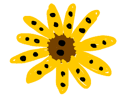
NEW: True for $r \sim p \log(pn)$ [Alweiss, Lovett, Wu, Zhang '19]

ROBUST LEMMA [Rossman'14, ALWZ'19, RaO'19, FKNP'19]

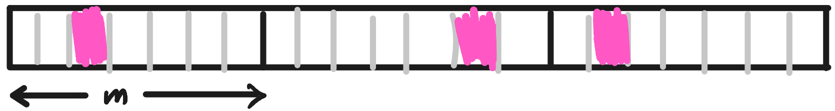
Let $\mathcal{X}$ be an n-uniform, block-respecting set system over $\mathcal{U} = \{x_1, \ldots, x_{mn}\}$

$\mathcal{U}$ = 

$m = 7$, $n = 3$

ROBUST  LEMMA

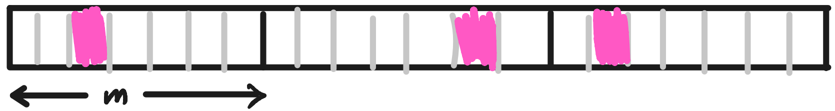
Let $\mathcal{X}$ be an n-uniform, block-respecting set system over $\mathcal{U} = \{x_1, \dots, x_{mn}\}$

$\mathcal{U} =$ 

$m = 7$, $n = 3$

ROBUST LEMMA

Let $\mathcal{X}$ be an n-uniform, block-respecting set system over $\mathcal{U} = \{x_1, \dots, x_{mn}\}$

$\mathcal{U} =$ 

$\mathcal{X}$ is **r-dense** if: $\forall I \subseteq [n]$ $H_\infty(\mathcal{X}_I) \geq r \cdot |I|$, $H_\infty(\mathcal{X}) = \min_{x \in \mathcal{X}} \log\left(\frac{1}{\Pr[\mathcal{X}=x]}\right)$

$$\mathcal{X}_{DNF} \stackrel{d}{=} x_3 x_{13} x_{16} \vee x_1 x_{10} x_{17} \vee x_3 x_8 x_{18} \vee x_7 x_{13} x_{20}$$

Theorem [ALWS] Let $\mathcal{X}$ be r-dense, $r \geq c \log(\frac{n}{\varepsilon})$. Then:

$$\Pr_{\rho \sim \{0,1\}^{mn}} [\mathcal{X}_{DNF}(\rho) \neq 1] \leq \varepsilon$$

Parameters: $m = n^{10}$, $r = .9 \log m$, $\varepsilon = 2^{-n^4}$

Simulation (Protocol $\Pi \to$ Decision tree T)

Invariant : $X \times Y \subseteq [m]^N \times \{0,1\}^{mN}$
X is $.9\log m$-dense
Y large : $|Y| \geq 2^{mN - N^2}$

- Initially (at root of Π), $X = [m]^N$, $Y = \{0,1\}^{mN}$
- When Bob sends a bit, go to larger side
- When Alice sends a bit, go to larger side

  If X no longer $.9\log m$ dense :

  - Find maximal subset $I \subseteq [N]$ and value $\alpha \in [m]^I$ that is too likely.
  - Query variables $z_I = \{x_i, i \in I\}$ in T. Say $z_I = \beta$
  - This induces a refinement of $X \times Y$:

$$X' = \{x \in X \mid x_I = \alpha\}$$
$$Y'_\beta = \{y \in Y \mid \mathrm{IND}(\alpha, y_I) = \beta\}$$

Need to show:
1. X' is dense
2. $\forall \beta \in \{0,1\}^{|I|}$ Y'_β is large

Simulation ($\text{Protocol } \Pi \to \text{Decision tree } T$)

Invariant : $X \times Y \subseteq [m]^N \times \{0,1\}^{mN}$
X is $.9\log m$-dense
Y large : $|Y| \geq 2^{mN - N^2}$

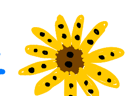
- Initially (at root of Π), $X = [m]^N$, $Y = \{0,1\}^{mN}$
- When Bob sends a bit, go to larger side
- When Alice sends a bit, go to larger side

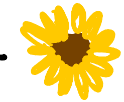
If X no longer $.9\log m$ dense :

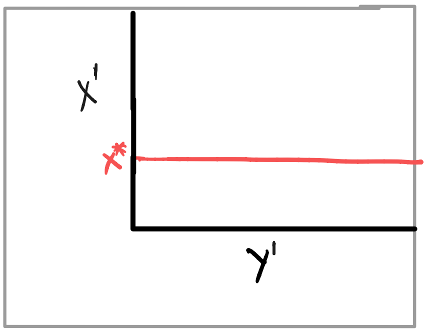
- Find maximal subset $I \subseteq [N]$ and value $\alpha \in [m]^I$ that is too likely.
- Query variables $z_I = \{x_i, i \in I\}$ in T. Say $z_I = \beta$
- This induces a refinement of $X \times Y$:

$$X' = \{x \in X \mid x_I = \alpha\}$$
$$Y'_\beta = \{y \in Y \mid \mathrm{IND}(\alpha, y_I) = \beta\}$$

Need to show:
1. X' is dense
2. $\forall \beta \in \{0,1\}^{|I|}$ Y'_β is large

Proof via Robust  Lemma

FULL RANGE LEMMA (via )

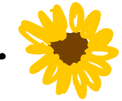


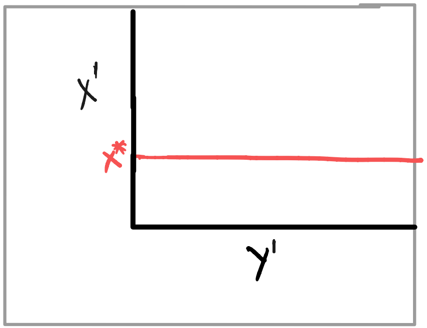
Let $X' \subseteq [m]^N$ be $.9 \log m$-dense

$Y' \subseteq \{0,1\}^{mN}$ be large

Then $\exists x^* \in X \quad \forall \beta \in \{0,1\}^N \quad \exists y^* \in Y'$

$\mathrm{IND}^N(x^*, y^*) = \beta$

FULL RANGE LEMMA (via )



Let $X' \subseteq [m]^N$ be $.9 \log m$-dense
$Y' \subseteq \{0,1\}^{mN}$ be large

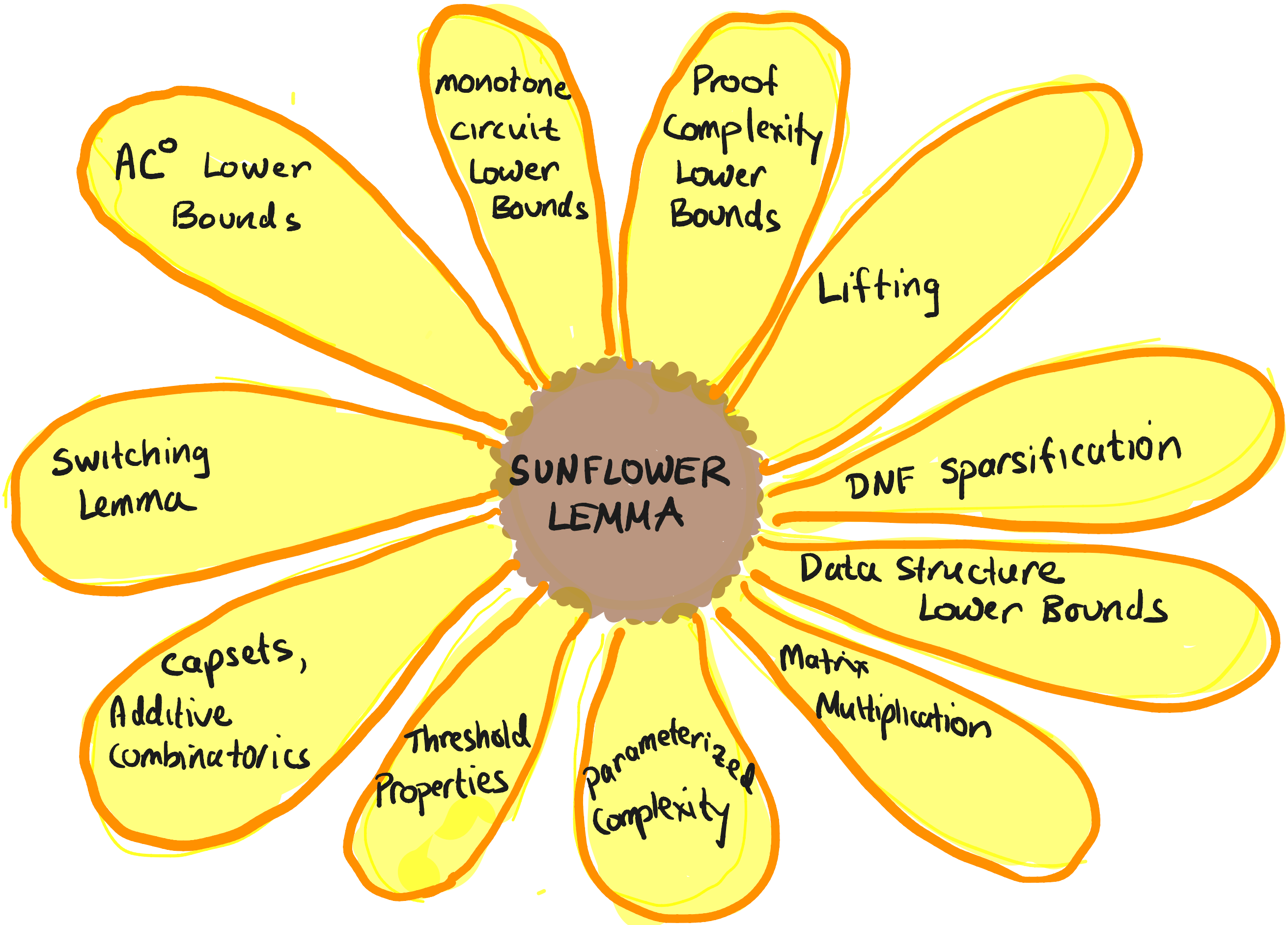
Then $\exists x \in X' \ \forall \beta \in \{0,1\}^N \ \exists y \in Y'$
$\mathrm{IND}^N(x, y) = \beta$

Proof sketch

If false $\forall x \in X' \ \exists \beta_x \in \{0,1\}^N \ \forall y \in Y' \ \mathrm{IND}^N(x^*, y^*) \neq \beta_x$

Can assume wlog that $\beta_x = 1^N \ \forall x$.

By Robust  Lemma, at most 2^{-N^4} fraction of all $y \in \{0,1\}^{mN}$ are bad which contradicts Y' largeness.


SUNFLOWER LEMMA
monotone circuit Lower Bounds
Proof Complexity Lower Bounds
Lifting
DNF sparsification
Data Structure Lower Bounds
Matrix Multiplication
parameterized complexity
Threshold Properties
capsets, Additive Combinatorics
Switching Lemma
AC^0 Lower Bounds

I. BEYOND MONOTONE LOWER BOUNDS ?

$P \stackrel{?}{=} NC^1$
$P \stackrel{?}{=} NC$

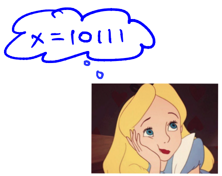
→ monotone lower bounds for slice functions

→ AC^0 circuits

→ KRW conjecture

I. BEYOND MONOTONE LOWER BOUNDS : AC^0

- Truly exponential size AC^0 LBs $\longrightarrow$ Formula size LBs
- KW for AC^0_d : cc of d-round protocols for KW_F equals $\log(AC^0_d\text{-size}(F))$



- Topdown via Lifting / Sunflower Lemma ?

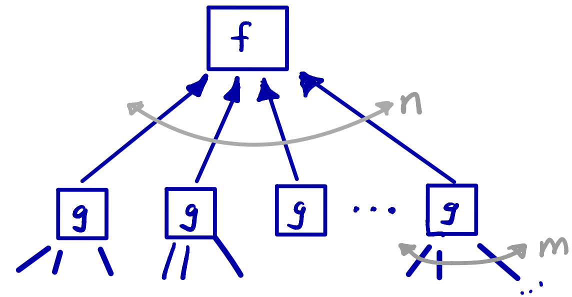
  ✓ $d = 3$ [Hastad, Jukna, Pudlak '95]

I. BEYOND MONOTONE LOWER BOUNDS: KRW

$f : \{0,1\}^n \to \{0,1\}$

$g : \{0,1\}^m \to \{0,1\}$

$\rightsquigarrow \quad f \circ g^n :$



$$\boxed{\textbf{KRW CONJECTURE}: \ \forall f, g \quad \mathrm{Depth}(f \circ g^n) \approx \mathrm{Depth}(f) + \mathrm{Depth}(g)}$$

Implies $P \neq NC^1$!

Prove: $CC(KW_{f \circ g^n}) \approx CC(KW_f) + CC(KW_g)$

I. BEYOND MONOTONE LOWER BOUNDS: KRW

$$CC(KW_{f \circ g^n}) \overset{?}{\approx} CC(KW_f) + CC(KW_g)$$

Long line of work resolving special cases:

[KRW'95, EIRS'01, HW'93, H&S98, DM'18, KM'18]

Theorem [de Rezende, Meir, Nordström, P, Robere '20]

① monotone KRW: $\forall$ monotone f, g $CC(mKW_{f \circ g^n}) \geq CC(KW_f) + CC(KW_g)$

solved for all lifted g

② "Semi-monotone" KRW

II. ALGEBRAIC CIRCUIT LOWER BOUNDS

- LOWER BOUNDS VIA CC/LIFTING?

- [Hrubeš]: monotone algebraic circuit LBs for ε-approx poly $\Rightarrow$ nonmonotone LBs
  Prove: $F_n + \varepsilon H_n$ hard for monotone circuits $\varepsilon < 2^{-n}$
  easy poly

Theorem [Chattopadyay, Datta, Mukhopadhyay '21]

Lower bounds for ε-approximate monotone for $\varepsilon \geq 2^{-\delta n}$

➡ Hard polynomials are lifted (SINK ∘ XOR)

➡ Proof is a reduction to discrepancy/corruption

III. LOGRANK CONJECTURE

$$\boxed{\forall f \quad CC(f) \stackrel{?}{=} \mathrm{poly}(\log \mathrm{rank}(M_f))}$$ [Lovasz, Saks '88]

UPPER BOUND: $CC(f) = \tilde{O}\left(\sqrt{\mathrm{rank}(M_f)}\right)$ [Lovett '16]

LOWER BOUND: $CC(f) = \Omega(\log^2 \mathrm{rank}(M_f))$ [Göös, P, Watson '17]

III. LOGRANK CONJECTURE

$$\boxed{\forall f \quad CC(f) \overset{?}{=} \mathrm{poly}(\log \mathrm{rank}(M_f))}$$

UPPER BOUND: $CC(f) = \tilde{O}\left(\sqrt{\mathrm{rank}(M_f)}\right)$ [Lovett '16]

LOWER BOUND: $CC(f) = \Omega\left(\log^2 \mathrm{rank}(M_f)\right)$ [Göös, P, Watson '17]

LOGRANK CONJECTURE FOR LIFTED FUNCTIONS?

Theorem [Knop, Lovett, McGuire, Yuan '21]

$$\forall f \quad \wedge\text{-DT}(f) = \mathrm{poly}(\log \mathrm{rank}(M_{f \circ \wedge^n})) \cdot \log n$$

$$\therefore \quad CC(f \circ \wedge^n) = \mathrm{poly}(\log \mathrm{rank}(M_{f \circ \wedge^n})) \cdot \log n$$

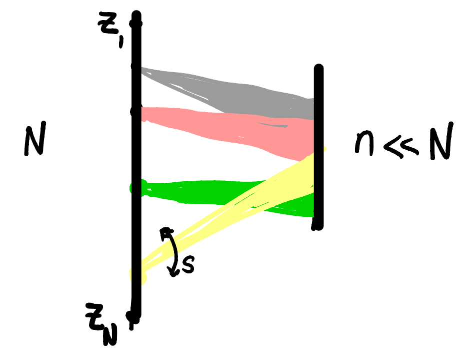
IV. SUPERCRITICAL SIZE-DEPTH TRADEOFFS

Theorem [Fleming, P, Robere '21] Let P = Resolution or Cutting Planes

There exist UNSAT FORMULAS F_n over n variables that have size 2^n, depth n P-refutations but any P-refutation of size $< 2^{n^\delta}$ requires depth $\Omega(2^{n^\delta})$

Proof Uses Composition to shrink the number of variables while preserving depth/size [Razborov]

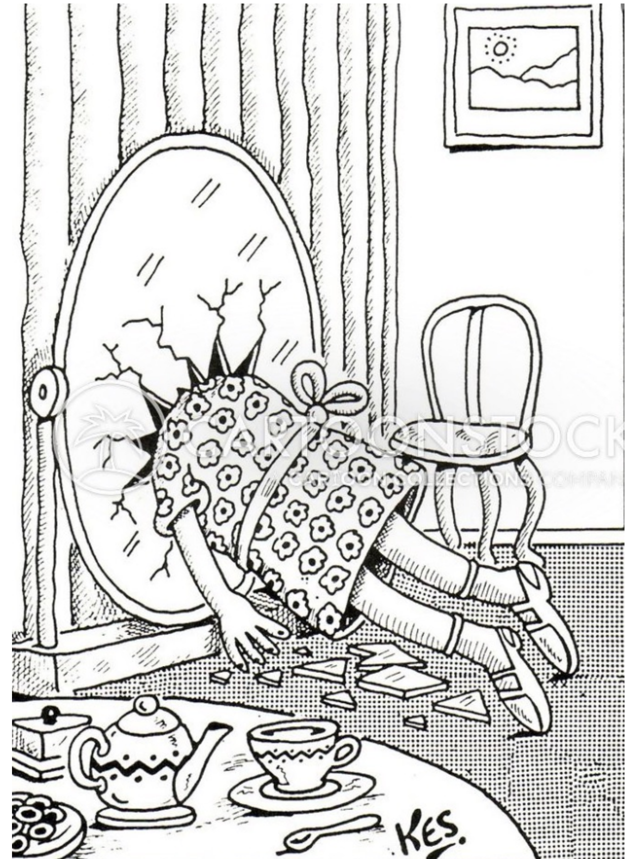
f_N over N vars $\rightsquigarrow$ $F_n = f_N \circ \oplus^s$



More Open Questions

- Lifting with constant gadget size?
- Dag-Lifting : randomized
  other gadgets (inner product)
- Supercritical tradeoffs for monotone circuits?
- AC^0 Lower Bounds via Lifting
- Other models : pseudodeterministic
  NOF
  Information complexity
- Other Applications : Algebraic circuits
  Data Structure Lower Bounds
  Combinatorics ...

Thanks!



LIFTING THEOREMS

Class	Query	Communication	References
P	deterministic DT	deterministic	RM'99 GPW'18
NP	Nondet DT	Nondet	GLMTZ'16
BPP	randomized DT	randomized	GPW'17 CFKMP'21
	polynomial degree	rank	S'11 SZ'09
PLS	Resolution	dag-like CC	GGKS'18, LMMPZ'21
PPA	Nullstellensatz	Algebraic Tiling	PR'17 PR'18
	Sherali-Adams SA	LP Extension Compl.	CLRS'16
	Sum-of Squares SOS	SDP Extension Compl.	LRS'15